MATH71000, Spring 2024 **Renato Ghini Bettiol**

Final Exam

Due: May 17, 2024

1. Consider the Lie group

$$\mathsf{G} = \left\{ \begin{pmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{GL}(3,\mathbb{R}) : x > 0,\ y, z \in \mathbb{R} \right\},$$

whose Lie algebra, endowed with the Lie bracket $[A,B] = AB - BA$ from $\mathfrak{gl}(3,\mathbb{R})$, is

$$\mathfrak{g} = \operatorname{span}\left\{ X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \subset \mathfrak{gl}(3,\mathbb{R}).$$

a) Compute the Lie brackets $[X,Y]$, $[Y,Z]$, $[X,Z]$ of the basis elements of $\mathfrak{g}$, and write them in terms of X, Y, Z.

b) Let g be the left-invariant Riemannian metric on G that at $\mathrm{Id} \in \mathsf{G}$ coincides with the inner product on $\mathfrak{g}$ for which $\{X, Y, Z\}$ is an orthonormal basis. Denote by $X, Y, Z \in \mathfrak{X}(\mathsf{G})$ the left-invariant vector fields corresponding to $X, Y, Z \in \mathfrak{g}$. Use the Koszul formula and a) to compute the the Levi-Civita connection ∇ of (G, g),

$$\begin{array}{ccc} \nabla_X X = & \nabla_X Y = & \nabla_X Z = \\ \nabla_Y X = & \nabla_Y Y = & \nabla_Y Z = \\ \nabla_Z X = & \nabla_Z Y = & \nabla_Z Z = \end{array}$$

(To avoid unnecessary computations, recall that $\nabla_A B - \nabla_B A = [A,B]$.)

c) Show that the curvature operator $R\colon \wedge^2 \mathfrak{g} \to \wedge^2 \mathfrak{g}$ is $R = -\mathrm{Id}$. Conclude that g is not a bi-invariant metric, and that (G, g) is isometric to hyperbolic 3-space.

d) On a matrix Lie group, such as G, the adjoint action is given by conjugation, i.e., $\mathrm{Ad}(A)B = ABA^{-1}$, for all $A \in \mathsf{G}$ and $B \in \mathfrak{g}$. Show that

$$\text{if } A = \begin{pmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ then } \begin{array}{l} \mathrm{Ad}(A)X = X - yY - zZ, \\ \mathrm{Ad}(A)Y = xY, \\ \mathrm{Ad}(A)Z = xZ, \end{array}$$

and use the matrix that represents $\mathrm{Ad}(A)\colon \mathfrak{g} \to \mathfrak{g}$ in the basis $\{X, Y, Z\}$ to compute the eigenvalues of $\mathrm{Ad}(A)$. Conclude that G does not admit bi-invariant metrics.

a) The Lie brackets (given by commutators) can be computed as follows:

$$[X,Y] = Y, \quad [X,Z] = Z, \quad [Y,Z] = 0$$

b) Since X, Y, Z and g are left-invariant, terms of the form $X(\mathrm{g}(Y,Z))$ vanish. Thus, Koszul's formula simplifies to the last three terms, which are determined by the above brackets. Moreover, since $\{X, Y, Z\}$ is orthonormal, we can skip computing some instances of Koszul's formula, e.g.,

$$0 = X(\mathrm{g}(X,X)) = 2\mathrm{g}(\nabla_X X, X), \quad \text{hence} \quad \nabla_X X = aY + bZ$$
$$0 = X(\mathrm{g}(X,Y)) = \mathrm{g}(\nabla_X X, Y) + \mathrm{g}(X, \nabla_X Y) \quad \text{hence} \quad a = -\mathrm{g}(X, \nabla_X Y)$$
$$0 = X(\mathrm{g}(X,Z)) = \mathrm{g}(\nabla_X X, Z) + \mathrm{g}(X, \nabla_X Z) \quad \text{hence} \quad b = -\mathrm{g}(X, \nabla_X Z).$$

Altogether, we find:

$$\begin{array}{lll} \nabla_X X = 0 & \nabla_X Y = 0 & \nabla_X Z = 0 \\ \nabla_Y X = -Y & \nabla_Y Y = X & \nabla_Y Z = 0 \\ \nabla_Z X = -Z & \nabla_Z Y = 0 & \nabla_Z Z = X \end{array}$$

c) The curvature operator $R\colon \wedge^2\mathfrak{g} \to \wedge^2\mathfrak{g}$ is diagonal on the basis $\{X\wedge Y, X\wedge Z, Y\wedge Z\}$ and all eigenvalues are equal to -1. For instance, we compute:

$$\begin{aligned} \langle R(X\wedge Y), X\wedge Y\rangle &= \langle \nabla_X\nabla_Y Y - \nabla_Y\nabla_X Y - \nabla_{[X,Y]}Y, X\rangle \\ &= \langle \nabla_X X - \nabla_Y Y, X\rangle \\ &= -1 \\ \langle R(X\wedge Y), X\wedge Z\rangle &= \langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, X\rangle \\ &= \langle -\nabla_Y Z, X\rangle \\ &= 0 \\ \langle R(X\wedge Y), Y\wedge Z\rangle &= \langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, Y\rangle \\ &= \langle -\nabla_Y Z, Y\rangle \\ &= 0, \end{aligned}$$

the remaining (similar) computations needed are:

$$\begin{aligned} \langle R(X\wedge Z), X\wedge Z\rangle &= -1 \\ \langle R(Y\wedge Z), Y\wedge Z\rangle &= -1 \\ \langle R(X\wedge Z), Y\wedge Z\rangle &= 0. \end{aligned}$$

Thus, (G, g) has $\sec \equiv -1$, so g is not bi-invariant (for otherwise it would have $\sec \geq 0$). As G is simply-connected, it follows that (G, g) is isometric to $\mathbb{H}^3$.

d) Since $\mathrm{Ad}(A)B = ABA^{-1}$, we compute

$$\begin{aligned} \mathrm{Ad}(A)X &= \begin{pmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{x} & -\frac{y}{x} & -\frac{z}{x} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -y & -z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = X - yY - zZ \end{aligned}$$

$$\mathrm{Ad}(A)Y = \begin{pmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{x} & -\frac{y}{x} & -\frac{z}{x} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = xY$$

$$\mathrm{Ad}(A)Z = \begin{pmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{x} & -\frac{y}{x} & -\frac{z}{x} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = xZ.$$

Thus, $\mathrm{Ad}(A)$ is represented in the basis $\{X, Y, Z\}$ by the upper triangular matrix

$$\mathrm{Ad}(A) = \begin{pmatrix} 1 & 0 & 0 \\ -y & x & 0 \\ -z & 0 & x \end{pmatrix},$$

whose eigenvalues are $1, x, x$. All inner products on $\mathfrak{g}$ are of the form $\mathrm{g}(P\cdot, \cdot)$ where $P\colon \mathfrak{g} \to \mathfrak{g}$ is positive-definite and g-symmetric ($P^T = P$), so $\mathrm{Ad}(A)$ is a linear isometry of such an inner product if and only if $(\mathrm{Ad}(A))^T P \,\mathrm{Ad}(A) = P$, i.e., $\mathrm{Ad}(A)^{-1} = P^{-1}(\mathrm{Ad}(A))^T P$. If this is case, $\mathrm{Ad}(A)^{-1}$ has the same eigenvalues as $\mathrm{Ad}(A)$, but that is only possible if $x = 1$. Thus, $\mathrm{Ad}(A)$ cannot be a linear isometry of $\mathfrak{g}$ for all $x > 0$, regardless of the inner product chosen on $\mathfrak{g}$, so G does not admit a bi-invariant metric.

2. Let (M^4, g) be a closed Riemannian 4-manifold. The conformal metric $\mathrm{h} = u^2\,\mathrm{g}$, where $u\colon M^4 \to \mathbb{R}$ is a positive smooth function, has scalar curvature given by:

$$\mathrm{scal}_{\mathrm{h}} = \big(-6\,\Delta u + u\,\mathrm{scal}_{\mathrm{g}}\big)\,u^{-3},$$

where $\Delta u = \operatorname{tr}\operatorname{Hess} u$. Suppose $\mathrm{scal}_{\mathrm{g}} \equiv \kappa$ and $\mathrm{scal}_{\mathrm{h}} \equiv \kappa$ are both constant and equal.

a) Prove that if $\kappa \neq 0$, then either $u \equiv 1$, or there exist points $p, q \in M$ such that $u(p) < 1 < u(q)$.

b) Prove that if $\kappa < 0$, then $\mathrm{h} = \mathrm{g}$. What happens if $\kappa = 0$?

a) As $\mathrm{scal}_{\mathrm{g}} \equiv \kappa$ and $\mathrm{scal}_{\mathrm{h}} \equiv \kappa$, we have that $u\colon M \to \mathbb{R}$ solves the PDE

$$\Delta u = \tfrac{\kappa}{6}\,u\,(1 - u^2).$$

Note that if a positive solution $u\colon M \to \mathbb{R}$ is constant, then $\Delta u = 0$, so $u \equiv 1$. Since M is closed, by Stokes' Theorem, we have that integrating both sides

$$0 = \int_M \Delta u \,\mathrm{vol}_{\mathrm{g}} = \tfrac{\kappa}{6} \int_M u\,(1-u^2)\,\mathrm{vol}_{\mathrm{g}}.$$

Thus, $1-u^2$ has zero weighted average on M with respect to the measure $u\,\mathrm{vol}_{\mathrm{g}}$, so either $u(x) \geq 1$ for all $x \in M$ or $u(x) \leq 1$ for all $x \in M$ imply $u \equiv 1$. Therefore, if $u \not\equiv 1$, then there exist $p, q \in M$ such that $u(p) < 1 < u(q)$.

b) Since M is closed, $u\colon M \to \mathbb{R}$ achieves a minimum p_0 and a maximum q_0. At the minimum, $0 \leq \Delta u(p_0) = \frac{\kappa}{6}\, u(p_0)\,(1-u(p_0)^2)$. By a), if $u \not\equiv 1$, then $0 < u(p_0) < 1$, so the previous inequality would imply that $\kappa \geq 0$. Since we have $\kappa < 0$, it follows that $u \equiv 1$, hence $\mathrm{h} = \mathrm{g}$.

If $\kappa = 0$, then the PDE becomes $\Delta u = 0$, whose only (positive) solutions on the closed manifold M are (positive) constants $u \equiv c$. Thus, in that case, instead of $\mathrm{h} = \mathrm{g}$, the conclusion becomes $\mathrm{h} = c^2\,\mathrm{g}$ for some constant $c > 0$, i.e., h and g are homothetic.

3. Let K^2 be the Klein bottle, and recall that it is double-covered by the 2-torus T^2. Provide either a construction (just a brief outline of the curvature computations is fine) or a topological obstruction (quoting a theorem) as answer to the following questions:

a) Does $K^2 \times S^1$ admit a Riemannian metric with $\sec \leq 0$? How about $\sec < 0$?

b) Does $K^2 \times \mathbb{R}P^2$ admit a Riemannian metric with $\mathrm{Ric} > 0$? How about $\mathrm{Ric} \geq 0$?

c) Does $\mathbb{C}P^n$ admit a Riemannian metric with $\sec \leq 0$? How about $\sec \geq 10$?

d) Does $\mathbb{S}^3/\mathbb{Z}_3 \times \mathbb{S}^3/\mathbb{Z}_5$ admit a metric with $\sec > 0$? How about $\mathrm{scal} \equiv k > 0$?

a) The double covering $T^3 \to K^2 \times S^1$ can be endowed with a flat metric such that the group $\mathbb{Z}_2$ of deck transformations act as isometries, thus $K^2 \times S^1$ admits a flat metric; i.e., with $\sec \equiv 0$, in particular, with $\sec \leq 0$. By Preissmann's Theorem, it does not admit a metric with $\sec < 0$, since $\pi_1(K^2 \times S^1) \cong (\mathbb{Z} \rtimes \mathbb{Z}) \times \mathbb{Z}$ contains Abelian subgroups nonisomorphic to $\mathbb{Z}$, e.g., $\mathbb{Z} \oplus \mathbb{Z}$.

b) By Myers' Theorem, $K^2 \times \mathbb{R}P^2$ does not admit a metric with $\mathrm{Ric} > 0$, since, by compactness, it would have $\mathrm{Ric} > k > 0$, while $\pi_1(K^2 \times \mathbb{R}P^2) \cong (\mathbb{Z} \rtimes \mathbb{Z}) \times \mathbb{Z}_2$ is not finite, or, equivalently, the universal covering $\mathbb{R}^2 \times \mathbb{S}^2$ is not compact. It does admit metrics with $\mathrm{Ric} \geq 0$, e.g., the product metric of the flat metric on K^2 and the round metric on $\mathbb{R}P^2$ with $\sec \equiv 1$ has $\sec \geq 0$; in particular, $\mathrm{Ric} \geq 0$.

c) By the Cartan–Hadamard Theorem, there is no metric with $\sec \leq 0$ on $\mathbb{C}P^n$, since its universal covering is $\mathbb{C}P^n$ itself, which is not diffeomorphic to $\mathbb{R}^{2n}$. The Fubini–Study metric g on $\mathbb{C}P^n$ has $\sec_{\mathrm{g}} \geq 1$, so $\frac{1}{10}\mathrm{g}$ has $\sec_{\frac{1}{10}\mathrm{g}} = 10\sec_{\mathrm{g}} \geq 10$.

d) By Synge's Theorem, the fundamental group of an even-dimensional closed manifold with $\sec > 0$ is either $\mathbb{Z}_2$ or trivial, so $\mathbb{S}^3/\mathbb{Z}_3 \times \mathbb{S}^3/\mathbb{Z}_5$ does not admit a metric with $\sec > 0$, as it has $\pi_1(\mathbb{S}^3/\mathbb{Z}_3 \times \mathbb{S}^3/\mathbb{Z}_5) \cong \mathbb{Z}_{15}$. Each of $\mathbb{S}^3/\mathbb{Z}_3$ and $\mathbb{S}^3/\mathbb{Z}_5$ admit metrics with $\sec \equiv 1$, hence $\mathrm{scal} \equiv 6$, so the product metric has $\mathrm{scal} \equiv 12 > 0$.

4. Let (M, g) be a connected compact Riemannian manifold with $\sec_M \geq 0$ and compact boundary $\partial M \neq \emptyset$. Suppose ∂M is convex, i.e., for all $p \in \partial M$, the shape operator $(S_{\vec{n}})_p \colon T_p\partial M \to T_p\partial M$, given by $S_{\vec{n}}(X) = -\nabla_X \vec{n}$, is positive-definite, where $\vec{n}$ is the inward-pointing unit normal of ∂M.

a) Does ∂M have to be connected? If yes, give a proof; if no, give a counter-example.

b) Prove that $\pi_1(M, \partial M) = \{1\}$, hence the inclusion $\partial M \hookrightarrow M$ induces a surjective homomorphism $\pi_1(\partial M) \to \pi_1(M)$. You may use (without proof) the fact that minimizing length in a nontrivial free homotopy class of curves in M with endpoints in ∂M yields a geodesic in M with endpoints in ∂M.

c) Give two examples of the above situation, to show that $\pi_1(\partial M) \to \pi_1(M)$ may or may not be injective.

a) Yes, ∂M has to be connected.
Suppose $\partial M = \bigcup_{j=1}^k N_j$ has multiple connected components. As ∂M is compact, let $p, q \in \partial M$ be the closest points among pairs of points in different connected components of ∂M, say $p \in N_1$ and $q \in N_2$. Since (M, g) is complete, there exists a minimizing unit speed geodesic $\gamma \colon [0, L] \to M$ with $\gamma(0) = p$ and $\gamma(L) = q$ such that $\mathrm{dist}(x, y) \geq L$ for all $x \in N_1$ and $y \in N_2$.
Given $v \in T_pN_1 = T_p\partial M$, let $V(t) = P_t v$ be the parallel transport of v along $\gamma(t)$. Since γ minimizes length from N_1 to N_2, by the first variation of energy, we have $\dot\gamma(0) = \vec{n}$ and $\dot\gamma(L) = -\vec{n}$, so $V(L) \in T_qN_2$. Let $\gamma_s(t)$, $t \in [0, L]$, $s \in (-\varepsilon, \varepsilon)$, be a variation of γ with endpoints in ∂M and variational field V. In order to facilitate the computation in the second variation formula, it is convenient to choose γ_s in such way that $\alpha \colon (-\varepsilon, \varepsilon) \to N_1$, $\alpha(s) = \gamma_s(0)$, and $\beta \colon (-\varepsilon, \varepsilon) \to N_2$, $\beta(s) = \gamma_s(L)$, are geodesics in N_1 and N_2 respectively, i.e., $\nabla^{N_1}_{\dot\alpha}\dot\alpha = 0$ and $\nabla^{N_2}_{\dot\beta}\dot\beta = 0$. Note this choice is possible as we only prescribed the variational field $V = \frac{\mathrm{d}}{\mathrm{d}}\gamma_s|_{s=0}$, so the endpoints $\alpha(s) = \gamma_s(0)$ and $\beta(s) = \gamma_s(L)$ can be chosen as any curves tangent to $V(0)$ and $V(L)$, e.g., $\alpha(s) = \exp_p^{N_1}(sV(0))$ and $\beta(s) = \exp_q^{N_2}(sV(L))$. Then,

$$\mathrm{g}\big(\tfrac{DV}{\mathrm{d}s}, \dot\gamma\big)\big|_{t=0} = \mathrm{g}\big(\nabla^M_{\dot\alpha}\dot\alpha, \dot\gamma(0)\big) = \mathrm{g}\big(\nabla^{\partial M}_{\dot\alpha}\dot\alpha + \mathrm{I\!I}_{\partial M}(\dot\alpha, \dot\alpha), \vec{n}\big) = \mathrm{g}(S_{\vec{n}}(\dot\alpha), \dot\alpha) > 0$$
$$\mathrm{g}\big(\tfrac{DV}{\mathrm{d}s}, \dot\gamma\big)\big|_{t=L} = \mathrm{g}\big(\nabla^M_{\dot\beta}\dot\beta, \dot\gamma(L)\big) = \mathrm{g}\big(\nabla^{\partial M}_{\dot\beta}\dot\beta + \mathrm{I\!I}_{\partial M}(\dot\beta, \dot\beta), -\vec{n}\big) = -\mathrm{g}(S_{\vec{n}}(\dot\beta), \dot\beta) < 0,$$

since $S_{\vec{n}}$ is positive-definite. Thus, by the second variation formula,

$$\tfrac{\mathrm{d}^2}{\mathrm{d}s^2}E_{\mathrm{g}}(\gamma_s)\big|_{s=0} = \mathrm{g}\big(\tfrac{DV}{\mathrm{d}s}, \dot\gamma\big)\Big|_0^L + \int_0^L \mathrm{g}\left(\tfrac{DV}{\mathrm{d}t}, \tfrac{DV}{\mathrm{d}t}\right) - \mathrm{g}(R(V, \dot\gamma)\dot\gamma, V)\,\mathrm{d}t < 0,$$

so for sufficiently small $0 < |s| < \varepsilon$, the curve γ_s, is shorter than $\gamma_0 = \gamma$ and joins N_1 to N_2, contradicting the choice of γ. Thus, ∂M is connected.

b) Suppose $\pi_1(M, \partial M) \neq \{1\}$, and let $\gamma\colon [0, L] \to M$ be a geodesic in M with endpoints $\gamma(0), \gamma(L) \in \partial M$ obtained by minimizing length on a nontrivial free homotopy class of curves in M with endpoints in ∂M. As in the previous item, by the first variation formula, $\dot\gamma(0) = \vec{n}$ and $\dot\gamma(L) = -\vec{n}$, and an identical computation with the second variation formula contradicts the minimality of γ. Thus, $\pi_1(M, \partial M) = \{1\}$ and hence the long exact sequence

$$\cdots \to \pi_1(\partial M) \to \pi_1(M) \to \pi_1(M, \partial M) \to \pi_0(\partial M) = \{1\}$$

implies that the induced homomorphism $\pi_1(\partial M) \to \pi_1(M)$ is surjective.

c) Let $M^n \subset \mathbb{R}^n$ be a strictly convex body with smooth boundary. Endowed with the flat metric induced from $\mathbb{R}^n$, we have $\sec_M \equiv 0$ and $\partial M \cong \mathbb{S}^{n-1}$ has $S_{\vec{n}} \succ 0$, cf. HW4 Prob. 2 and X. If $n = 2$, then $\pi_1(\partial M) \cong \mathbb{Z} \to \pi_1(M) \cong \{1\}$ is not injective, while for $n \geq 3$ it is trivially injective since $\pi_1(\partial M) \cong \pi_1(M) \cong \{1\}$.

Note that the conclusions above remain valid if the hypotheses $\sec_M \geq 0$ and $S_{\vec{n}} \succ 0$ are weakend to $\mathrm{Ric}_M \succeq 0$ and $H_{\partial M} = \operatorname{tr} S_{\vec{n}} > 0$, by replacing v with an orthonormal basis $v_i \in T_p\partial M$, and summing the second variation formula over $1 \leq i \leq n-1$.

5. Recall from Problem 6 in the Midterm that if (M, g) is a complete noncompact manifold and $p \in M$, then there exists a unit speed geodesic $\gamma\colon \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\mathrm{dist}(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \geq 0$. Assume, moreover, that (M^n, g) has $\mathrm{Ric} \geq 0$.

   a) Fix $a > 0$, and use Bishop Volume Comparison to show that, for all $t > a$,

$$\frac{\mathrm{Vol}\big(B_{t+a}(\gamma(t))\big)}{\mathrm{Vol}\big(B_{t-a}(\gamma(t))\big)} \leq \frac{(t+a)^n}{(t-a)^n}$$

   b) Show that $B_a(p) \subset B_{t+a}(\gamma(t)) \setminus B_{t-a}(\gamma(t))$, and conclude that, given $t > t_0 > a$,

$$\mathrm{Vol}\big(B_{t-a}(\gamma(t))\big) \geq c(n, t_0)\, \mathrm{Vol}(B_a(p))\, t,$$

   where $c(n, t_0) = \inf\limits_{t \in [t_0, +\infty)} \frac{1}{t} \frac{(t-a)^n}{(t+a)^n - (t-a)^n} > 0$.

   c) Show that $B_{\frac{r+a}{2} - a}\big(\gamma\big(\frac{r+a}{2}\big)\big) \subset B_r(p)$ and conclude that, for all $r > 2t_0 - a$,

$$c\, r \leq \mathrm{Vol}(B_r(p)) \leq C\, r^n,$$

   where $c, C > 0$ are constants. In particular, (M^n, g) has infinite volume.

   d) For each $1 \leq k \leq n$, give an example of a complete noncompact Riemannian manifold (M^n, g) with $\mathrm{Ric} \geq 0$ for which $\mathrm{Vol}(B_r(p)) = O(r^k)$ as $r \nearrow +\infty$.

a) Let $\overline{B_r} \subset \mathbb{R}^n$ be an Euclidean ball of radius r, and recall that $\mathrm{Vol}(\overline{B_r}) = \mathrm{Vol}(\overline{B_1})r^n$. By Bishop Volume Comparison, since (M^n, g) has $\mathrm{Ric} \geq 0$, the function

$$r \mapsto \frac{\mathrm{Vol}\left(B_r(\gamma(t))\right)}{\mathrm{Vol}\left(\overline{B_r}\right)}$$

is nonincreasing, thus, for $t > a$, we have

$$\frac{\mathrm{Vol}\left(B_{t-a}(\gamma(t))\right)}{\mathrm{Vol}\left(\overline{B_{t-a}}\right)} \geq \frac{\mathrm{Vol}\left(B_{t+a}(\gamma(t))\right)}{\mathrm{Vol}\left(\overline{B_{t+a}}\right)}$$

so

$$\frac{\mathrm{Vol}\left(B_{t+a}(\gamma(t))\right)}{\mathrm{Vol}\left(B_{t-a}(\gamma(t))\right)} \leq \frac{\mathrm{Vol}\left(\overline{B_{t+a}}\right)}{\mathrm{Vol}\left(\overline{B_{t-a}}\right)} = \frac{\mathrm{Vol}(\overline{B_1})(t+a)^n}{\mathrm{Vol}(\overline{B_1})(t-a)^n} = \frac{(t+a)^n}{(t-a)^n}.$$

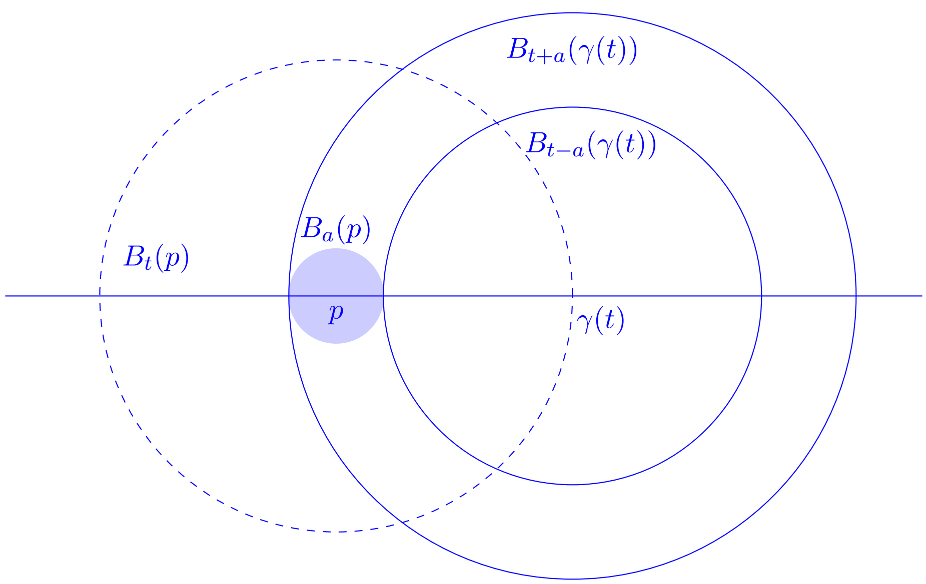
b) Using the triangle inequality, we have that if $x \in B_a(p)$, i.e., $\mathrm{dist}(x,p) < a$, then

$$\mathrm{dist}(x, \gamma(t)) \leq \mathrm{dist}(x,p) + \mathrm{dist}(p, \gamma(t)) < a + t,$$

so $x \in B_{t+a}(\gamma(t))$. However, if $x \in B_{t-a}(\gamma(t))$, then $\mathrm{dist}(x, \gamma(t)) < t - a$, so

$$t = \mathrm{dist}(p, \gamma(t)) \leq \mathrm{dist}(p, x) + \mathrm{dist}(x, \gamma(t)) < a + (t-a) = t,$$

which is a contradiction. Thus, we must have $x \notin B_{t-a}(\gamma(t))$, hence proving that $B_a(p) \subset B_{t+a}(\gamma(t)) \setminus B_{t-a}(\gamma(t))$, see figure below. Thus, for all $t > a$, using the



inequality from a), we have that

$$\frac{\mathrm{Vol}(B_a(p))}{\mathrm{Vol}(B_{t-a}(\gamma(t)))} \leq \frac{\mathrm{Vol}\left(B_{t+a}(\gamma(t))\right) - \mathrm{Vol}\left(B_{t-a}(\gamma(t))\right)}{\mathrm{Vol}(B_{t-a}(\gamma(t)))}$$

$$= \frac{\mathrm{Vol}\left(B_{t+a}(\gamma(t))\right)}{\mathrm{Vol}\left(B_{t-a}(\gamma(t))\right)} - 1 \leq \frac{(t+a)^n}{(t-a)^n} - 1,$$

and hence

$$\mathrm{Vol}\left(B_{t-a}(\gamma(t))\right) \geq \mathrm{Vol}(B_a(p))\,\frac{1}{\frac{(t+a)^n}{(t-a)^n}-1} = \mathrm{Vol}(B_a(p))\,\frac{(t-a)^n}{(t+a)^n-(t-a)^n}.$$

Given $t > t_0 > a$, setting $c(n,t_0) = \inf\limits_{t\in[t_0,+\infty)} \frac{1}{t}\frac{(t-a)^n}{(t+a)^n-(t-a)^n} > 0$, we obtain

$$\mathrm{Vol}\left(B_{t-a}(\gamma(t))\right) \geq \mathrm{Vol}(B_a(p))\,c(n,t_0)\,t.$$

c) If $\mathrm{dist}\big(x,\gamma\big(\frac{r+a}{2}\big)\big) < \frac{r+a}{2} - a$, then by the triangle inequality,

$$\mathrm{dist}(x,p) \leq \mathrm{dist}\big(x,\gamma\big(\tfrac{r+a}{2}\big)\big) + \mathrm{dist}\big(\gamma\big(\tfrac{r+a}{2}\big),p\big) < \big(\tfrac{r+a}{2}-a\big) + \tfrac{r+a}{2} = r,$$

so $x \in B_r(p)$, which proves that $B_{\frac{r+a}{2}-a}\big(\gamma\big(\frac{r+a}{2}\big)\big) \subset B_r(p)$. Setting $t = \frac{r+a}{2}$ in the inequality obtained in b), we have that if $t > t_0$, i.e., $r > 2t_0 - a$,

$$\mathrm{Vol}(B_r(p)) \geq \mathrm{Vol}\left(B_{\frac{r+a}{2}-a}\big(\gamma\big(\tfrac{r+a}{2}\big)\big)\right) \geq \mathrm{Vol}(B_a(p))\,c(n,t_0)\,\tfrac{r+a}{2} > c\,r$$

where $c = \frac{1}{2}\mathrm{Vol}(B_a(p))\,c(n,t_0)$. So there exist constants $c, C > 0$ such that

$$c\,r \leq \mathrm{Vol}(B_r(p)) \leq C\,r^n,$$

where $C = \mathrm{Vol}(\overline{B_1})$, once again by Bishop Volume Comparison. Since the above lower bound for $\mathrm{Vol}(B_r(p))$ holds for all $r > 2t_0 - a$, taking $r \nearrow +\infty$ we conclude that (M^n, g) has infinite volume.

d) For each $1 \leq k \leq n$, let $M^n = \mathbb{S}^{n-k} \times \mathbb{R}^k$ and g be the product metric, which has $\sec \geq 0$ and hence $\mathrm{Ric} \geq 0$. By Fubini's theorem, $\mathrm{Vol}(B_r(p))$ is asymptotic to $\mathrm{Vol}(\mathbb{S}^{n-k})\ \mathrm{Vol}(\overline{B_1})\,r^k$ where $\overline{B_1} \subset \mathbb{R}^k$ is the Euclidean unit ball.

Note: The above constant $c = \frac{1}{2}\mathrm{Vol}(B_a(p))\,c(n,t_0)$ cannot be made "universal", i.e., independent of $\mathrm{Vol}(B_a(p))$, as shown by examples of C. Croke of complete noncompact manifolds with $\mathrm{Ric} \geq 0$ and $\inf_{p\in M} \mathrm{Vol}(B_a(p)) = 0$.

6. Consider the unit sphere $\mathbb{S}^5 \subset \mathbb{R}^6$ and let $M = \mathbb{S}^5 \cap (\mathbb{R}^3 \oplus \{0\})$ and $N = \mathbb{S}^5 \cap (\{0\} \oplus \mathbb{R}^3)$, which are isometric copies of the unit sphere $\mathbb{S}^2$ sitting in $\mathbb{S}^5$.

a) Verify that M and N are totally geodesic in $\mathbb{S}^5$.

b) Given a unit vector $x \in \mathbb{R}^3$, identify $T_{(x,0)}M$ with a subspace of $T_{(x,0)}\mathbb{S}^5$, and let $T_{(x,0)}M^\perp \subset T_{(x,0)}\mathbb{S}^5$ be its orthogonal complement. If $v \in T_{(x,0)}M^\perp$ is a unit vector, find an explicit formula for the geodesic $\gamma(t) = \exp_{(x,0)} tv$ on $\mathbb{S}^5 \subset \mathbb{R}^6$.

c) Let $TM^\perp = \bigcup_{x\in M} T_{(x,0)}M^\perp$ be the normal bundle of $M \subset \mathbb{S}^5$, which is a trivial bundle $TM^\perp \cong M \times \mathbb{R}^3$. We write $(x,t,v) \in TM^\perp$, where $t \geq 0$ and $v \in T_{(x,0)}M^\perp$ is a unit vector. Show that the restriction of $f\colon TM^\perp \to \mathbb{S}^5$, $f(x,t,v) = \exp_{(x,0)} tv$, to $\{(x,t,v) \in TM^\perp : 0 \leq t < \frac{\pi}{2}\}$ is a diffeomorphism onto its image $\mathbb{S}^5 \setminus N$.

d) Show that, for each $(x,0) \in M$, the map $\phi_x \colon \mathbb{S}^2 \subset T_{(x,0)}M^\perp \to N$ given by $\phi_x(v) = f(x, \frac{\pi}{2}, v)$ is an isometry.

e) Show that $M, N \subset \mathbb{S}^5$ are subsets at maximal distance $\operatorname{dist}_g(M,N) = \pi/2$, i.e., the function $M \ni x \mapsto \operatorname{dist}_g(x,N) = \inf\{\operatorname{dist}_g(x,y) : y \in N\}$ is constant and equal to $\pi/2$, and for any subset $P \subset \mathbb{S}^5$, $P \not\subset N$, one has $\operatorname{dist}_g(x,P) < \frac{\pi}{2}$ for some $x \in M$.

a) The unique geodesic $\gamma \colon \mathbb{R} \to \mathbb{S}^5 \subset \mathbb{R}^6$ with initial conditions $\gamma(0) = p$ and $\dot\gamma(0) = v$ is $\gamma(t) = (\cos t)p + (\sin t)v$. Thus, if either $p \in M$ and $v \in T_pM$, i.e., $p = (p_1, 0)$ and $v = (v_1, 0)$, or if $p \in N$ and $v \in T_pN$, i.e., $p = (0, p_2)$ and $v = (0, v_2)$, then $\gamma(t) = \big((\cos t)p_1 + (\sin t)v_1, 0\big) \in M$ or $\gamma(t) = \big(0, (\cos t)p_1 + (\sin t)v_1\big) \in N$, respectively, for all $t \in M$, so M and N are totally geodesic.

b) By the same discussion above, the given geodesic $\gamma(t) = \exp_{(x,0)} tv$ on $\mathbb{S}^5 \subset \mathbb{R}^6$ is $\gamma(t) = \big((\cos t)x, (\sin t)v\big)$.

c) By the above, $f(x,t,v) = \exp_{(x,0)} tv = \big((\cos t)x, (\sin t)v\big)$, so its restriction to $M \times B_{\frac{\pi}{2}}(0) = \{(x,t,v) \in TM^\perp : 0 \leq t < \frac{\pi}{2}\}$ is a diffeomorphism onto its image $\mathbb{S}^5 \setminus N$, since $\cos t$ and $\sin t$ are diffeomorphisms from $(0, \frac{\pi}{2})$ to $(0,1)$; namely, its inverse is given by $f^{-1}(y,w) = \big(\frac{y}{\|y\|}, \arccos\|y\|, \frac{w}{\|w\|}\big)$ for all $(y,w) \in \mathbb{S}^5 \setminus (M \cup N)$ and $f^{-1}(y,0) = (y,0,0)$ if $(y,0) \in M$.

d) Identifying $T_{(x,0)}M^\perp \cong (\{0\} \oplus \mathbb{R}^3) \subset \mathbb{R}^6$, the map $\phi_x \colon \mathbb{S}^2 \subset T_{(x,0)}M^\perp \to N$ given by $\phi_x(v) = f(x, \frac{\pi}{2}, v) = (0, v)$ is the identity map, hence an isometry.

e) The geodesic $\gamma(t) = \exp_{(x,0)} tv = \big((\cos t)x, (\sin t)v\big)$ is minimizing for $t \in (0, \frac{\pi}{2})$, and $\gamma(t) \notin N$ for $t \in (0, \frac{\pi}{2})$, while $\exp_{(x,0)} \frac{\pi}{2} v = (0, v) \in N$, so for all $x \in M$, we have $\operatorname{dist}_g(x, N) = \frac{\pi}{2}$ and this distance is achieved along any geodesic normal to M. Alternatively, note that M and N are orbits of the isometric action of $\mathsf{SO}(3) \times \mathsf{SO}(3)$ on $\mathbb{S}^5 \subset \mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3$, hence these submanifolds are equidistant. Moreover, if $p = (p_1, p_2) \in \mathbb{S}^5 \setminus (M \cup N) \subset \mathbb{R}^6$, then $\operatorname{dist}(p, M) = \arccos\|p_1\| < \frac{\pi}{2}$ is achieved along a geodesic $\gamma(t) = \exp_{(x,0)} tv$, where $x = \frac{p_1}{\|p_1\|}$ and $v = \frac{p_2}{\|p_2\|} \in T_{(x,0)}M^\perp$, so N is the set at maximal distance from M, and vice-versa.