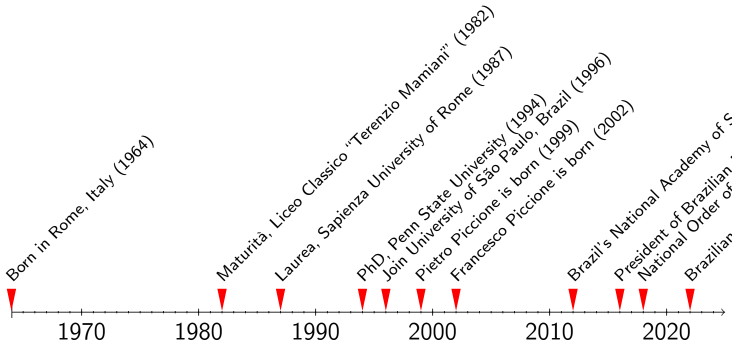
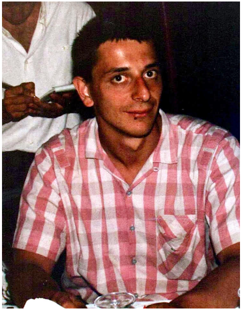
A 60-minute overview of the first 60 years of Paolo Piccione's mathematical career

THE CITY UNIVERSITY OF NEW YORK
Renato G Bettiol

Supported by NSF CAREER Award DMS-2142575


Born in Rome, Italy (1964)
Maturità, Liceo Classico "Terenzio Mamiani" (1982)
Laurea, Sapienza University of Rome (1987)
PhD, Penn State University (1994)
Join University of São Paulo, Brazil (1996)
Pietro Piccione is born (1999)
Francesco Piccione is born (2002)
Brazil's National Academy of Sciences (2012)
President of Brazilian Math. Soc. SBM (2017)
National Order of Scientific Merit (2018)
Brazilian citizenship (2022)
1970
1980
1990
2000
2010
2020




ICM 2018
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INTERNATIONAL CONGRESS OF MATHEMATICIANS
Firjan

Life in academia

- Research
- Teaching
- Service

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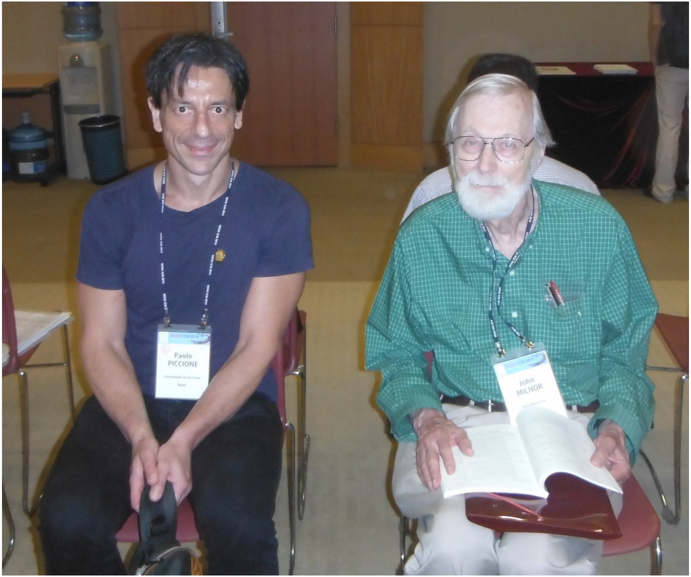
§1. Infinite-dimensional Morse Theory applied to geodesics



Geodesics

(M^n, g) *Lorentzian* manifold:

$$\mathrm{g} \sim \begin{pmatrix} - & & & & \\ & + & & & \\ & & \ddots & & \\ & & & + & \\ & & & & + \end{pmatrix}$$

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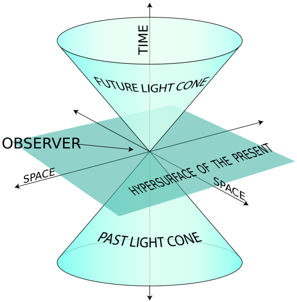

$$\mathrm{sign}(\mathrm{g}) = n_+(\mathrm{g}) - n_-(\mathrm{g})$$

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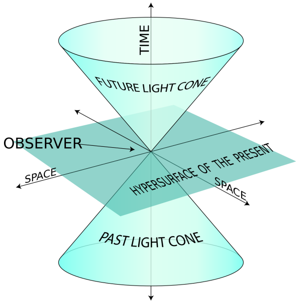

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$v \in T_pM$ is *spacelike* if $\mathrm{g}(v, v) > 0$
lightlike if $\mathrm{g}(v, v) = 0$
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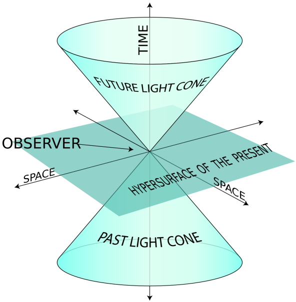

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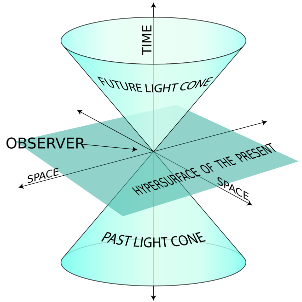

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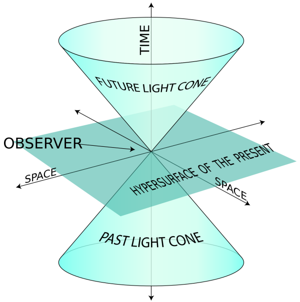

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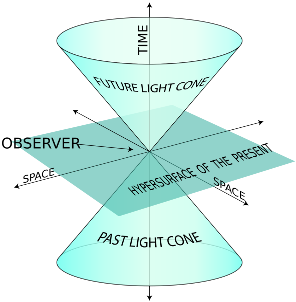

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[K. Uhlenbeck, 1975] Morse theory for (some) Lorentzian geodesics

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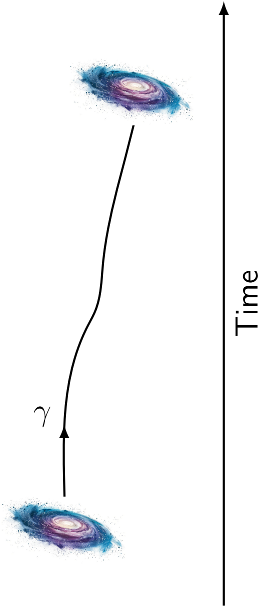
worldline of light source



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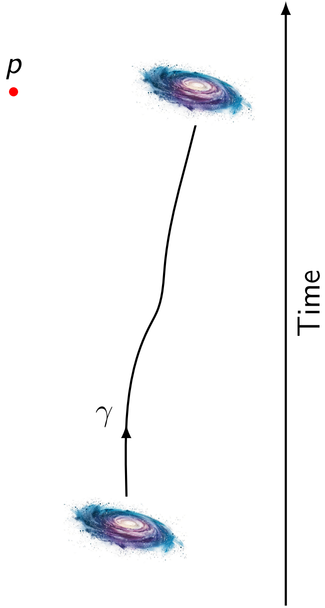
$p \in M$ event (observation)



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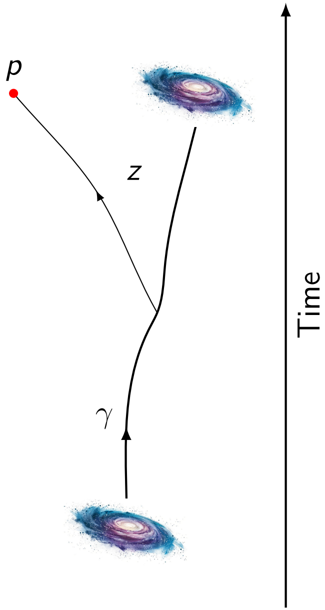

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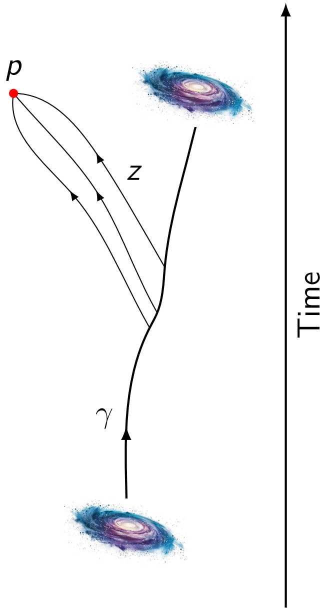

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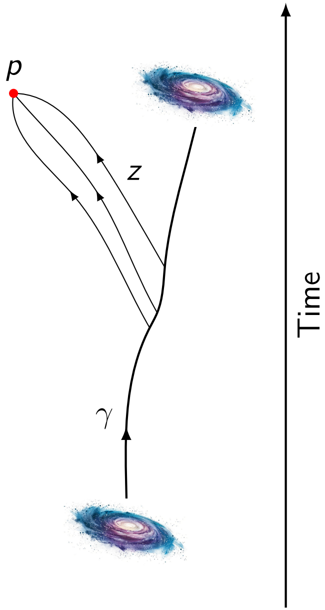
Multiple images?



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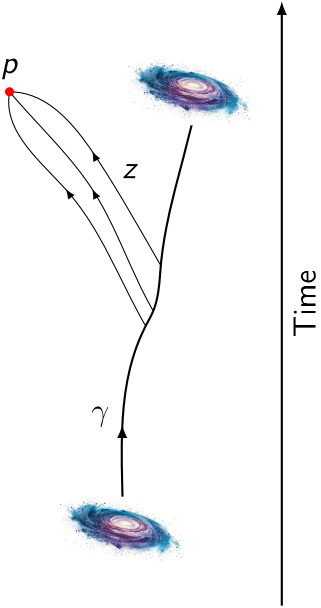
Gravitational Lensing

Galaxy PSZ1 G311.65-18.48



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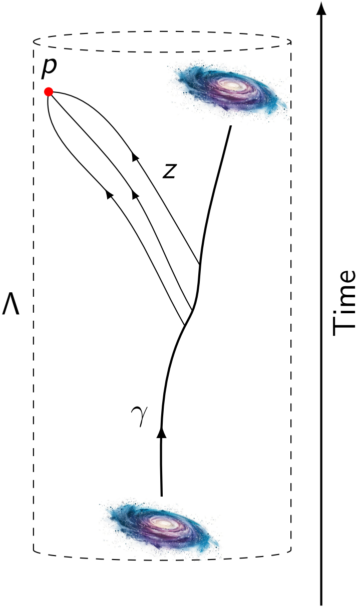

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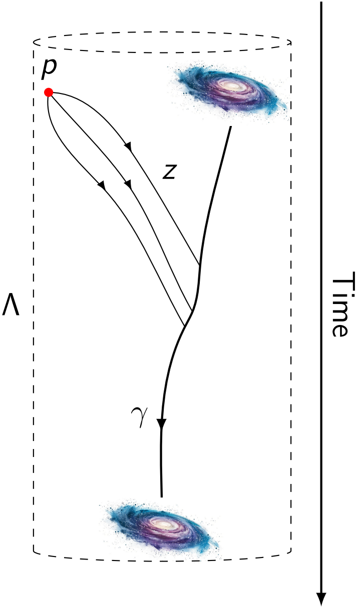

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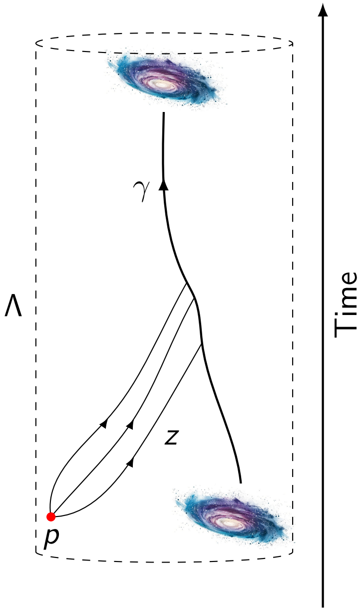

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Giannoni, Masiello, Piccione [CMP, 1997], [AIHP, 1998], [CVPDE, 1998]

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- Issue: $\mathcal{L}^+_{p,\gamma}(\Lambda)$ is not smooth

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- *Penalized* energy (to stay in Λ): $Q_\delta(z) := Q(z) + \delta \int_a^b \frac{\mathrm{d}s}{\mathrm{dist}(z,\partial\Lambda)}$

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The **Morse inequalities** *hold for Q, i.e., for any field $\mathbb{K}$,*

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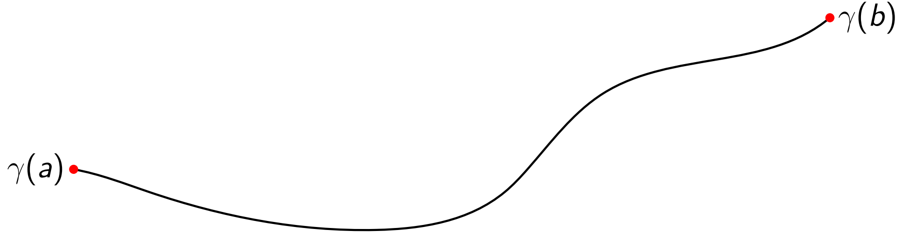
Conjugate points along geodesics



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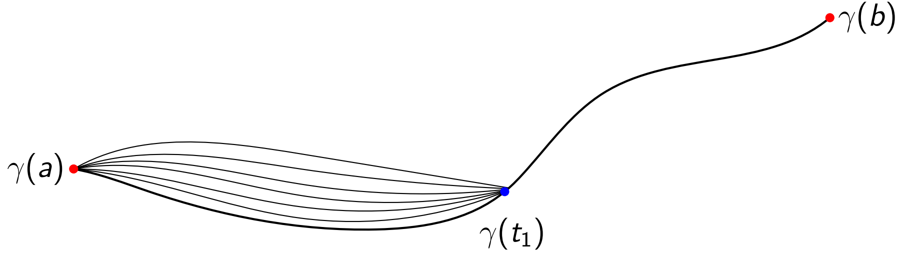

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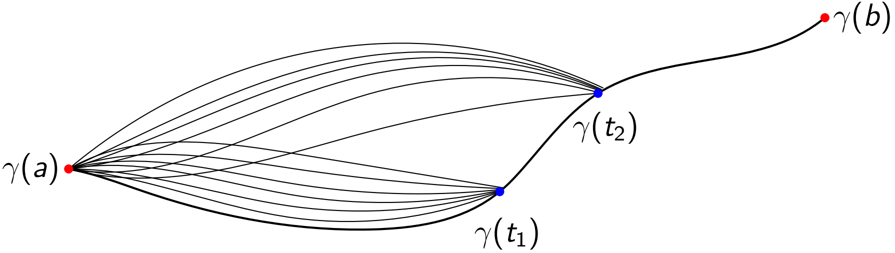

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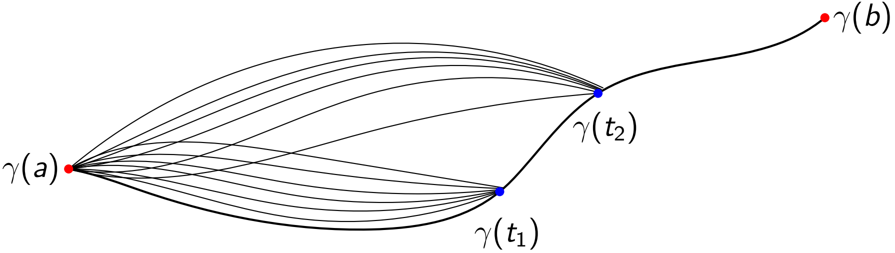

Theorem (Morse Index Theorem)

Given a **Riemannian geodesic** $\gamma \colon [a, b] \to M$ *with fixed endpoints,*

Morse Index of γ *as critical point of* $E = \#$ *conjugate points along* γ
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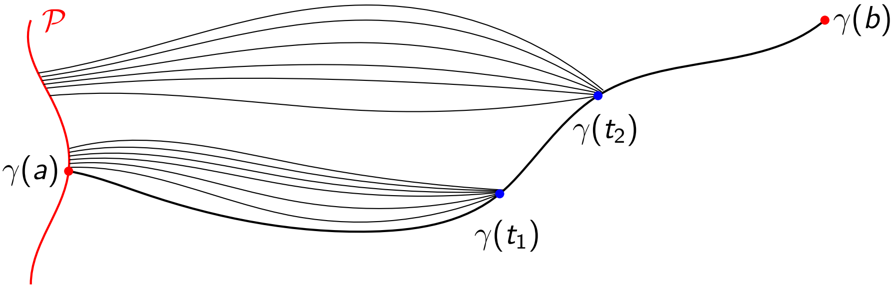

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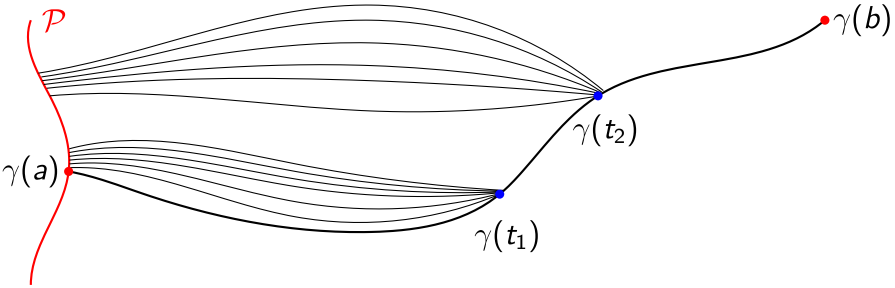

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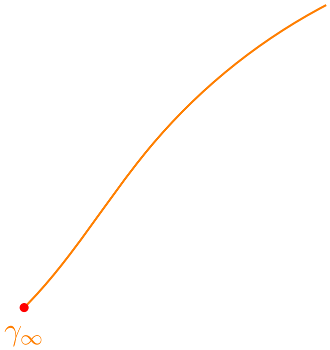
Morse theory in the limit?

- $\gamma_\infty \colon [a, b] \to M$ *lightlike* geodesic



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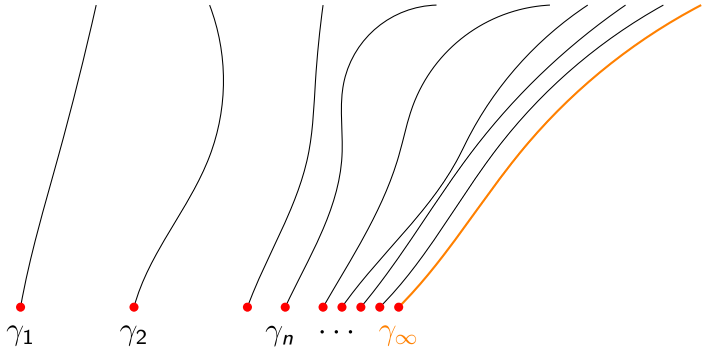

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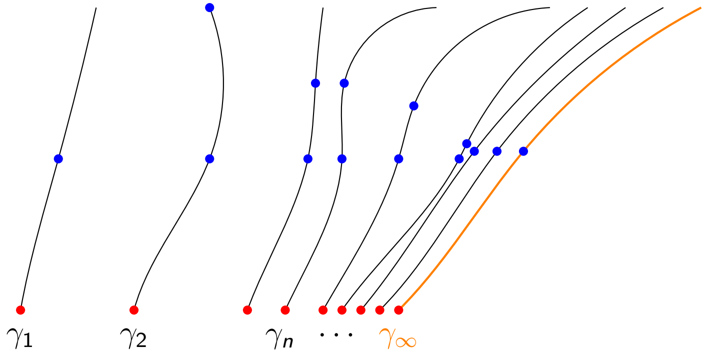

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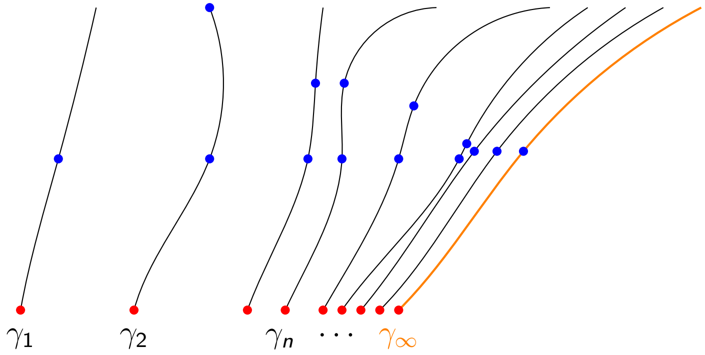

- Issue: index form $I_{\gamma_n}(\cdot,\cdot) = \mathrm{d}^2 E(\gamma_n)$ degenerates as $n \to \infty$!

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Francesco Mercuri (7 Jul 1946 – 5 Aug 2024)

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Something is preserved in the limit

- $\gamma\colon [a,b] \to M$ **semi-Riemannian** geodesic
- $\mathcal{J}[t] = \{ J(t) \mid J\colon [a,b] \to TM \text{ Jacobi field along } \gamma, J(a) = 0 \}$
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Theorem (Mercuri, Piccione, Tausk [PJM 2002])

If all conjugate points along γ are nondegenerate, then

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"algebraic count" of conjugate points

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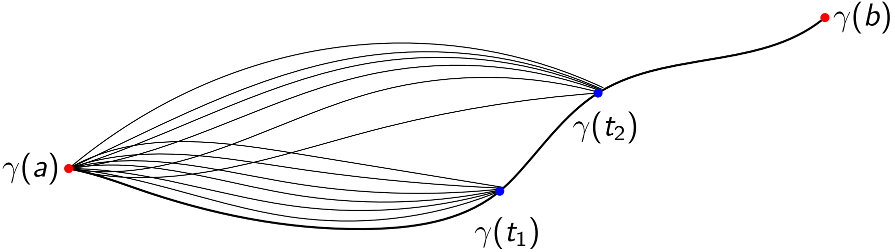
Homological invariant, stable under C^0 perturbations of γ

Degeneracies happen



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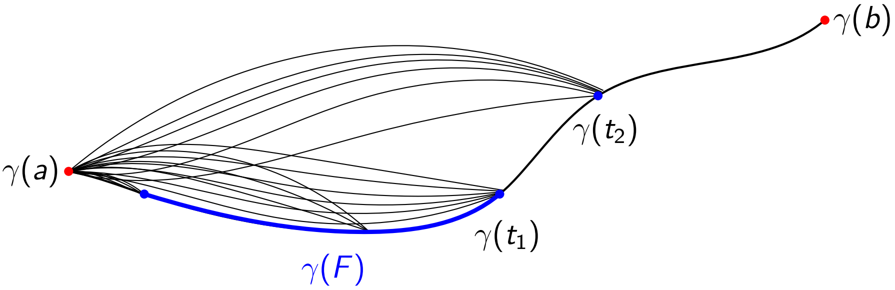

Theorem (Piccione, Tausk [CAG, 2003])

Given a closed subset $F \subset \mathbb{R}$ such that $F \subset (a, b]$, there is a **Lorentzian** *manifold (M, g) and a spacelike geodesic $\gamma\colon [a, b] \to M$ such that $\gamma(t)$ is conjugate to $\gamma(a)$ if and only if $t \in F$.*

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$\gamma \colon [a, b] \to M$ **semi-Riemannian** geodesic

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Theorem (Piccione, Tausk [Topology, 2002])

$$\textit{Maslov Index of } \gamma = n_-(I_\gamma|_\mathcal{K}) - n_+(I_\gamma|_\mathcal{S})$$

Other index theorems with Maslov index

Piccione, Tausk [Proc. LMS, 2001]

Index theorem for non-periodic solutions of **Hamiltonian systems**

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v \\ \alpha \end{pmatrix} = \underbrace{\begin{pmatrix} A(t) & B(t) \\ C(t) & -A^*(t) \end{pmatrix}}_{\in \mathfrak{sp}(2n,\mathbb{R})} \begin{pmatrix} v \\ \alpha \end{pmatrix}$$

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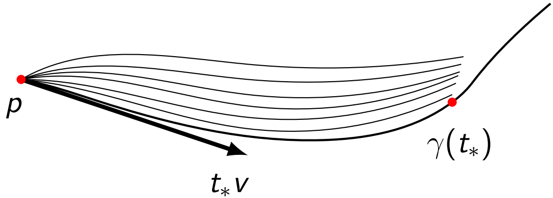
Index theorem for solutions of **constrained** variational problems
e.g., **sub-Riemannian** geodesics

Bifurcation of geodesics



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If $\gamma(t_*) = \exp_p t_* v$ *is conjugate to* p*,*

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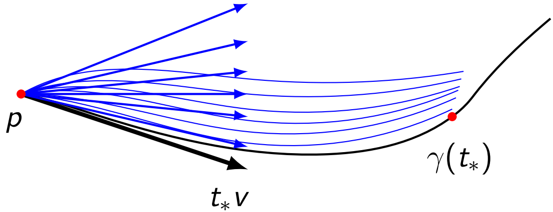

Theorem (Morse–Littauer, PNAS 1932)

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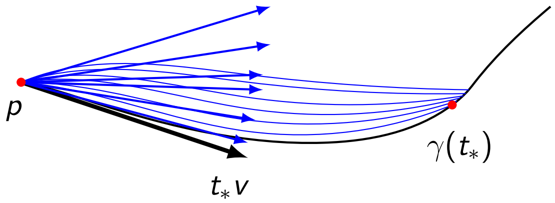

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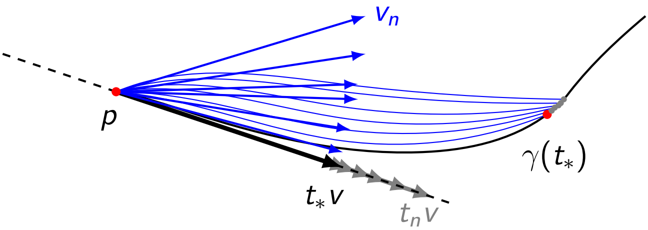

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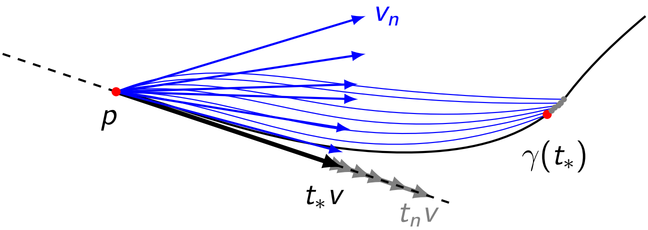

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Bifurcation: jump of Morse index $\Longrightarrow$ local nonuniqueness

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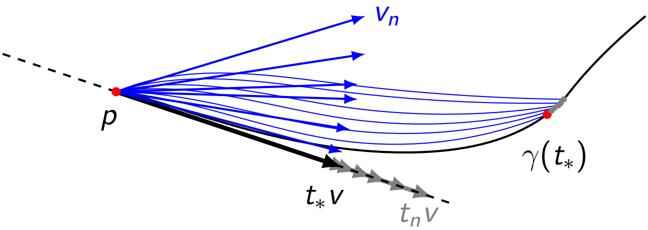

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Bifurcation: jump of *Maslov* index $\Longrightarrow$ local nonuniqueness

Piccione, Portaluri, Tausk [AGAG 2004]

Semi-Riemannian Morse–Littauer theorem

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Given $\mathcal{K}$, $\mathcal{K}_m$, etc. as above, there is a conformally flat Hilbert manifold with a geodesic γ having these conjugate points.

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§2. Bifurcation theory in Geometric Analysis

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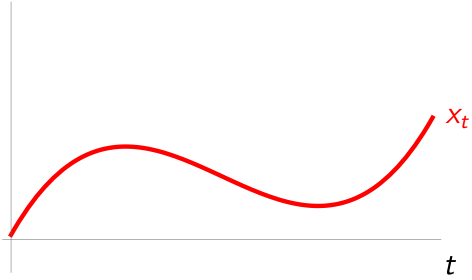
Bifurcation

x_t trivial branch

$$\mathrm{d}f_t(x_t) = 0$$



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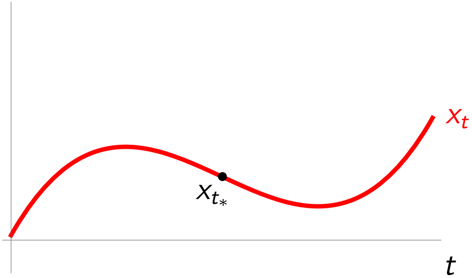

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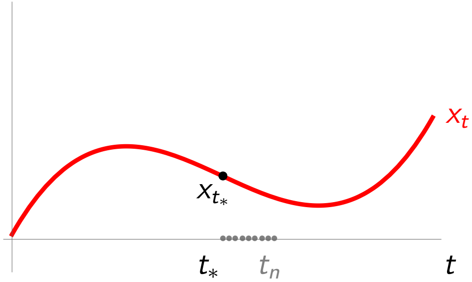

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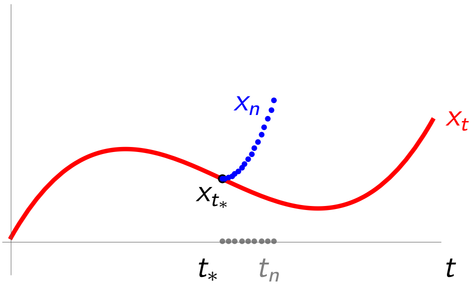

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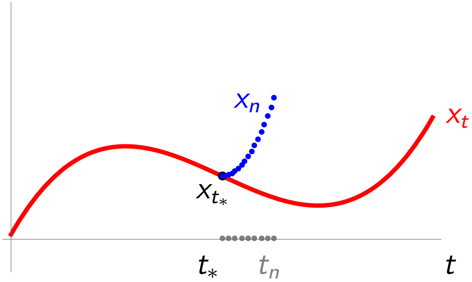

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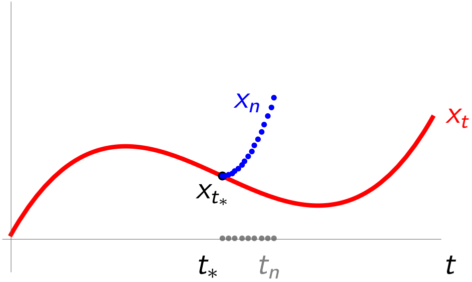

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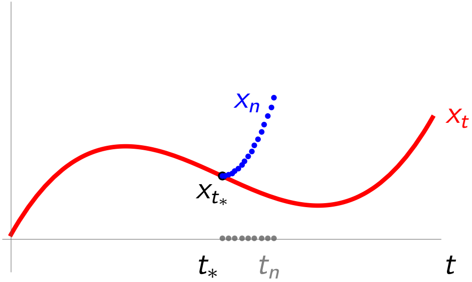

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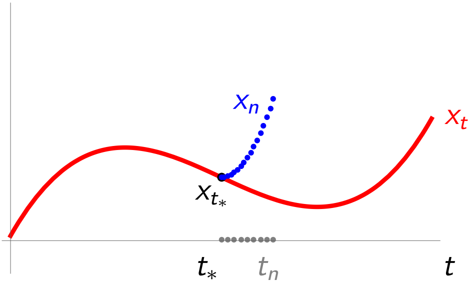

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$$\boxed{\text{Morse index jumps at } x_{t_*}} \implies \boxed{\text{bifurcation occurs at } x_{t_*}}$$

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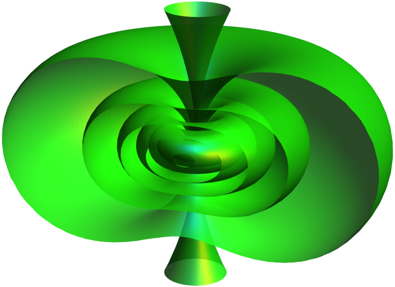

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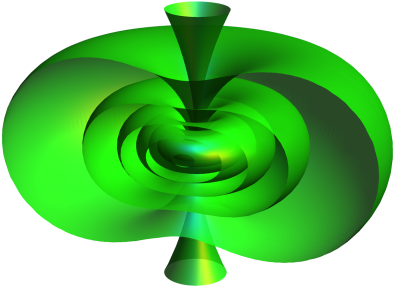

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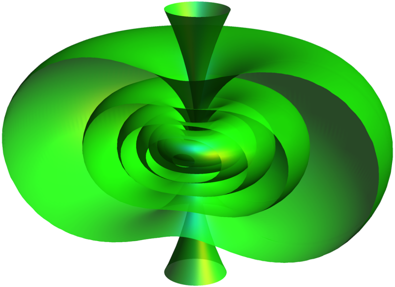

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For each $1 \leq k < n$*, there exist sequences accumulating at* 0 *and* $\frac{\pi}{2}$ *of bifurcations for the family* $x_t \colon \Sigma \hookrightarrow S^{n+1}$ *of CMC embeddings.*

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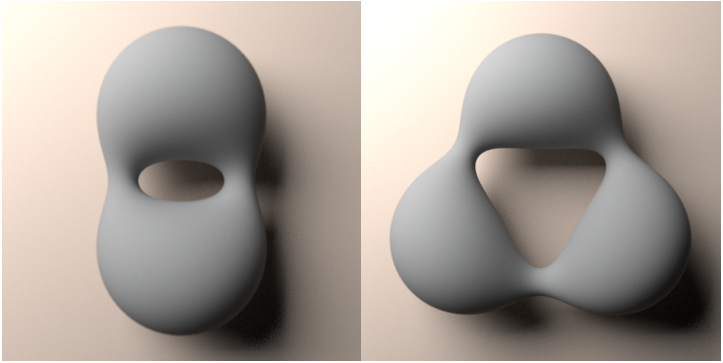

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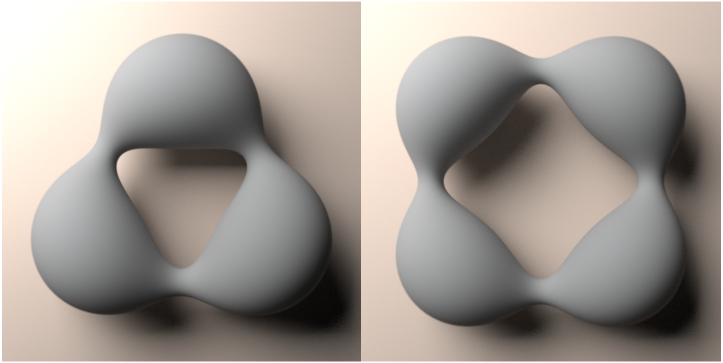

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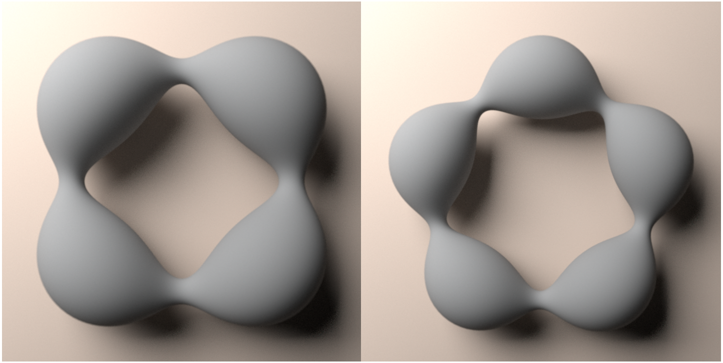

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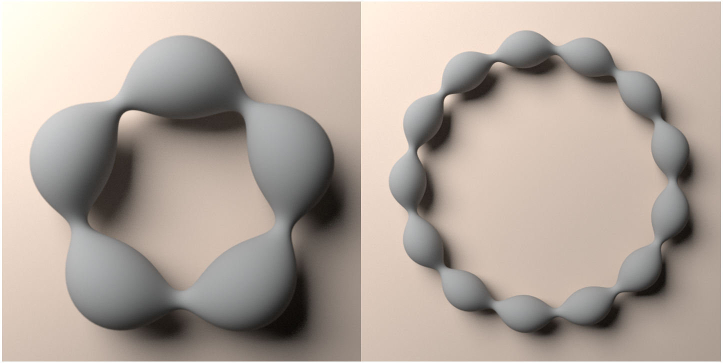

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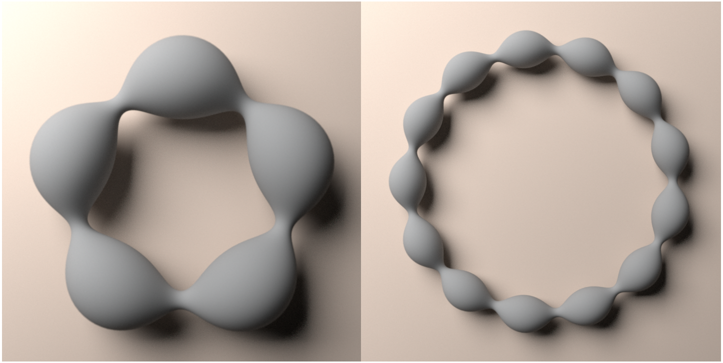

Morse index of $x_t = \#\,\mathrm{Spec}(-\Delta_{x_t(\Sigma)}) \cap \big(-\infty, \|A_{x_t(\Sigma)}\|^2 + \mathrm{Ric}(\vec{n}_{x_t})\big)$

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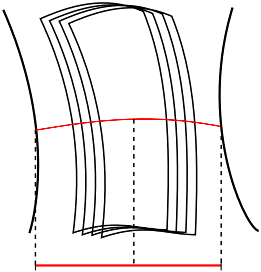
Theorem (B., Piccione [IMRN, 2016])

On a cohomogeneity one manifold with a normal isotropy subgroup, principal orbits bifurcate infinitely many times as they collapse to a singular orbit.



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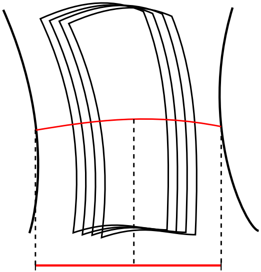

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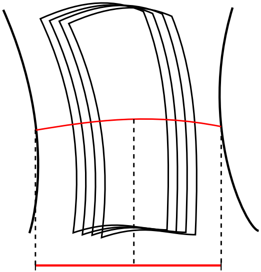

$\mathbb{C}P^n$, $\mathbb{H}P^n$,
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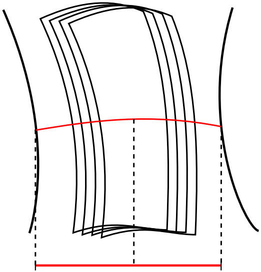

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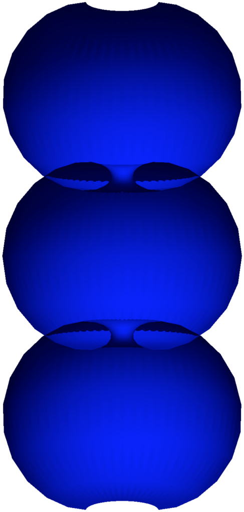
B., Lauret, Piccione [BLMS, 2022]

On $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{C}\mathrm{a}P^2$:

Computation of $\mathrm{Spec}(-\Delta_{x_t(\Sigma)}) \Longrightarrow$ explicit bifurcation instants

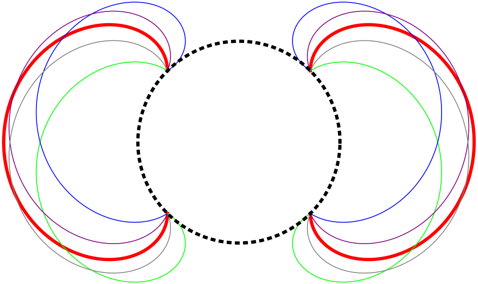
Constant Mean Curvature, IV

Nodoid in $\mathbb{R}^3$



Koiso, Palmer, Piccione [ACV, 2014]

There are infinitely many families of CMC surfaces in $\mathbb{R}^3$ with boundary on two fixed coaxial circles that bifurcate from portions of nodoids as their conormal angle varies.



Minimal surfaces

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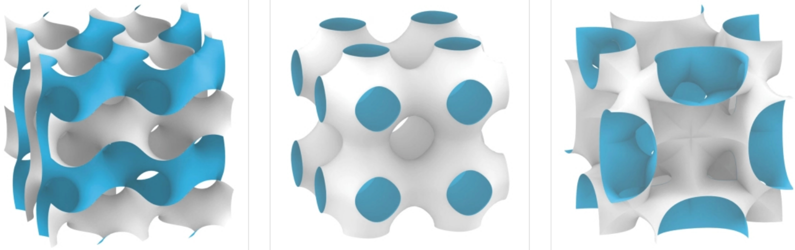

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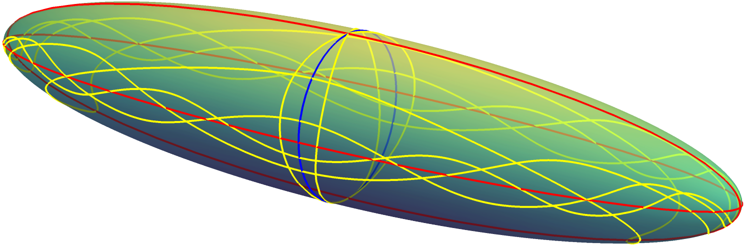

B., Piccione [Pisa, 2024]

Bifurcation of minimal 2-spheres in elongated 3-ellipsoids.

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Let $u_t \equiv 1/\,\mathrm{Vol}(M, \mathrm{g}_t)^{\frac{n-2}{2n}}$

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- (M^n, g_t) Riemannian manifolds, $n \geq 3$
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- $f_t(u) = \int_M u \, L_{\mathrm{g}_t} u \ \mathrm{vol}_{\mathrm{g}_t}, \qquad L_{\mathrm{g}_t} = -\Delta_{\mathrm{g}_t} + \frac{n-2}{4(n-1)} \mathrm{scal}_{\mathrm{g}_t}$
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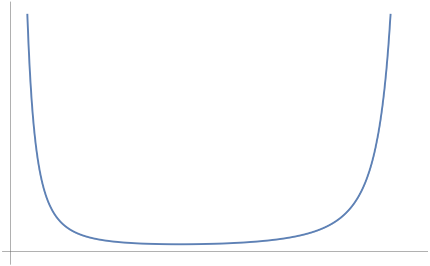

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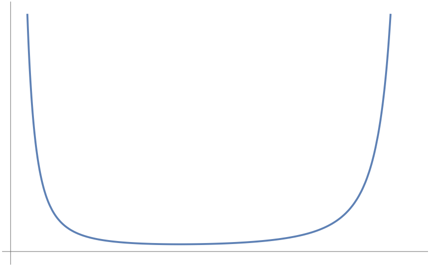

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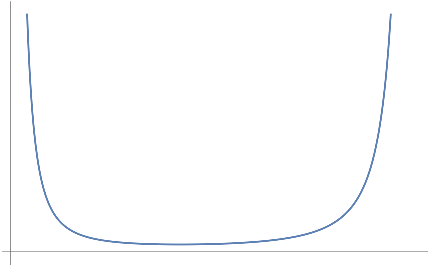

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Theorem (Lima, Piccione, Zedda [AIHP, 2012])

If (M_i, h_i) *have constant positive scalar curvature (and at least one is nondegenerate), then there are infinitely many bifurcation instants accumulating at* 0 *and* ∞ *for the Yamabe problem on*

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Collapse $\Longrightarrow$ Instability $\Longrightarrow$ Bifurcation

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Singular Yamabe Problem

B., Piccione, Santoro [JDG 2016]: See Bianca's talk tomorrow!

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B., Piccione, Sire [IMRN, 2021]: 4th order Q-curvature

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Andrade, Case, Piccione, Wei [2023]: GMJS operators

§3. Everything else

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Brake orbits

Giambò, Giannoni, Piccione [ARMA, 2011], [CAG, 2014], [CVPDE, 2015]

Brake orbits

Closed Lorentzian geodesics

Biliotti, Mercuri, Piccione [CAG, 2008]
Javaloyes, Lima, Piccione [Math Z, 2008]

Brake orbits

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Generic properties of semi-Riemannian geodesic flow

Giambò, Giannoni, Piccione [CMP, 2009]
Biliotti, Javaloyes, Piccione [IUMJ, 2009], [JLMS, 2011]

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B., Piccione, Siciliano [TG, 2014], [PNDE, 2014]

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Piccione, Zeghib [ETDS, 2014]

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B., Derdzinski, Piccione [AMPA, 2018]
B., Derdzinski, Mossa, Piccione [MNach, 2022]

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Spectral geometry
B., Lauret, Piccione [JGA, 2022], [BLMS, 2022]

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Spectral geometry

Isoperimetric problem, Allen–Cahn equation

Benci, Nardulli, Piccione [CVPDE, 2020]
Andrade, Conrado, Nardulli, Piccione, Resende [JFA, 2024]


Coca-Cola


zero
Coca-Cola
Coca-Cola

Happy birthday, Paolo!