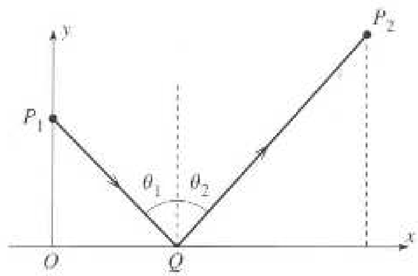
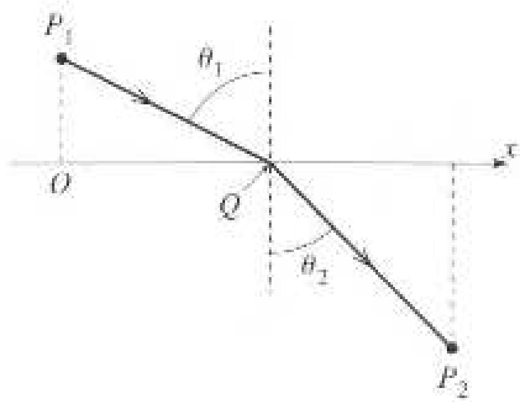
Prof. Anchordoqui

Problems set # 9 **Physics 411** November 1, 2007

1. Consider a ray of light traveling in vacuum from point P_1 to P_2 by way of the point Q on a plane miror as in the figure. Show that Fermat's principle implies that, on the actual path followed, Q lies in the same vertical plane as P_1 and P_2 and obeys the law of reflection, that $\theta_1 = \theta_2$. [Hints: Let the mirror lie in the xz plane, and let P_1 lie on the y-axis at $(0,\ y_1,\ 0)$ and P_2 in the xy plane at $(x_2, y_2, 0)$. Finally, let $Q = (x, 0, z)$. Calculate the time for the light to traverse the path P_1QP_2 and show that it is minimum when Q has $z = 0$ and satisfies the law of reflection.]



2. A ray of light travels from point P_1 in a medium of refractive index n_1 to P_2 in a medium of refractive index n_2, by way of the point Q on the plane interface between the two media as shown in the figure. Show that Fermat's principle implies that, on the actual path followed, Q lies in the same vertical plane as P_1 and P_2 and obeys Snell's law, that $n_1 \sin\theta_1 = n_2 \sin\theta_2$. [Hints: Let the interface be the xz plane, and let P_1 lie on the y axis at $(0, h_1, 0)$ and P_2 in the x, y plane at $(x_2, -h_2, 0)$. Finally, let $Q = (x, 0, z)$. Calculate the time for the light to traverse the path P_1QP_2 and show that it is a minimum when Q has $z = 0$ and satisfies Snell's law.]



3. The shortest path between to points on a curve surface, such as the surface of the sphere, is called a geodesic. To find a geodesic one has first to set up an integral that gives the length of a

path on the surface in question. *(i)* Use spherical polar coordinates (r, θ, ϕ) to show that the length of a path joining two points on a sphere of radius R is

$$L = R \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2 \theta \phi'^2(\theta)}\; d\theta \,,$$

if $(\theta_1,\ \phi_1)$ and $(\theta_2,\ \phi_2)$ specify the two points and we assume that the path is expressed as $\phi = \phi(\theta)$ *(ii)* Use the result in *(i)* to prove that the geodesic between two given points on the sphere is a great circle. [Hint: The integrand $f(\phi,\ \phi',\ \theta)$ is independent of ϕ so the Euler-Lagrange equation reduces to $\partial f/\partial \phi' = c$, a cosntant. This gives you ϕ' as a function of θ. You can avoid doing the final integral by using the following trick: there is no loss of generality in choosing your z-axis to pass through the point 1. Show that within this choice the constant c is necessarily zero, and describe the corresponding geodesic.]

4. Find and describe the path $y = y(x)$ for which the integral

$$\int_{x_1}^{x_2} \sqrt{x}\, \sqrt{1 + y'^2}\, dx$$

is stationary.

5. Find the path $y = y(x)$ for which the integral

$$\int_{x_1}^{x_2} x\, \sqrt{1 - y'^2}\, dx$$

is stationary.

6. In relativity theory, velocities can be represented by points in a certain rapidity space in which the distance between two neighboring points is

$$ds = \left[\frac{2}{1 - r^2}\right] \sqrt{dr^2 + r^2 d\phi^2},$$

where r are polar coordinates, and we consider just a two dimensional space. (an expression like this for the distance in a non-Euclidean space is often called metric of the space.) Use Euler-Lagrange equation to show that the shortest distance from the origin to any other point is a straight line.

7. You are given a string of fixed length ℓ with one end fastened at the origin O, and you are to place the string in the xy plane with its other end on the x axis in such a way as to enclose the maximum area between the string and the x axis. Show that the required shape is a semi-circle.

8. Prove that the shortest path between two points in three dimensions is a straight line.