PROBLEMS # 8

(1)

The equation of the orbit is

$$r = \frac{r_c}{1 - e\cos\theta}$$

where $r_c = \frac{L^2}{GM}$ and $e = \sqrt{1 + \frac{2EL^2}{GM}}$

Therefore the radial distance r can vary from the maximum value $\frac{r}{1-e}$ to the minimum value $\frac{r_c}{1+e}$.

Now the angular velocity of the particle is given by

$$\omega = \frac{L}{r^2}$$

Thus the maximum and minimum

Values of ω become

$$\omega_{max} = \frac{h}{r^2_{min}} = \frac{h}{\left[r_c/(1+e)\right]^2}$$

$$\omega_{min} = \frac{h}{r^2_{max}} = \frac{h}{\left[r_c/(1-e)\right]^2}$$

Then $$\frac{\omega_{max}}{\omega_{min}} = \left(\frac{1+e}{1-e}\right)^2 = n$$

from which we obtain

$$e = \frac{\sqrt{n} - 1}{\sqrt{n} + 1}$$

(2) Kepler's second law states that the AREAL velocity is constant, and this implies that the angular momentum per unit mass h is conserved. If a body is acted upon by a force and if the angular momentum of the body is not altered, then the force has imparted no torque on the body, thus the force must have acted on a along

the line connecting the force center and the body. That is the force is central.

Keplers first law states that planets move in elliptical orbits with the Sun at one of the focus. This means that the orbit can be described by

$$\frac{r_c}{r} = 1 - e\cos\theta \quad \text{with } 0 < e < 1 \quad (1)$$

On the other hand, for central forces

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{r^2}{h^2}\bar{F}(r) \quad (2)$$

where $\bar{F}(r)$ is the force per unit mass.

Substituting $1/r$ from (1) into the left hand side of (2) we find

$$\frac{1}{r_c} = -\frac{r^2}{h^2}\bar{F}(r) \Rightarrow \bar{F}(r) = -\frac{h^2}{r_c r^2}$$

now $r_c = \frac{l^2}{GM} \Rightarrow \vec{F}(r) = -\frac{GM}{r^2}$

$\Rightarrow$ the force $F(r) = m\bar{F}(r)$

$$F(r) = -\frac{GMm}{r^2}$$

③ $$\langle (a/r)^4 |\cos\theta| \rangle = \frac{1}{\tau}\int_0^\tau dt \left[\frac{1-e\cos\theta}{1-e^2}\right]^4 |\cos\theta|$$

$$a = \frac{r_c}{1-e^2} \qquad r = \frac{r_c}{1-e\cos\theta}$$

From Keppler's second law, we found the relation between t and θ

$$dt = \frac{\tau}{\pi ab}\, dA = \frac{\tau}{\pi ab}\frac{1}{2} r^2 d\theta$$

$$\Rightarrow \left\langle \left(\frac{a}{r}\right)^4 |\cos\theta| \right\rangle = \frac{1}{\tau}\frac{1}{(1-e^2)^4}\frac{\tau}{\pi ab}\frac{r_c^2}{2}\int_0^{2\pi} |\cos\theta|(1-e\cos\theta)^2 d\theta$$

It is easily seen that the value of the

integral is $2\pi e$ $\Rightarrow$

$$\left\langle \left(\frac{a}{r}\right)^4 |\cos\theta| \right\rangle = \frac{1}{(1-e^2)^4} \frac{1}{ab} r_c^2 e$$

Now $a = \dfrac{r_c}{1-e^2}$ $\qquad b = \dfrac{r_c}{\sqrt{1-e^2}}$

$$\Rightarrow \left\langle \left(\frac{a}{r}\right)^4 |\cos\theta| \right\rangle = \frac{e}{(1-e^2)^{5/2}}$$

(4) Start with the equation of the orbit $\dfrac{r_c}{r} = 1 - e\cos\theta$

and take its time derivative

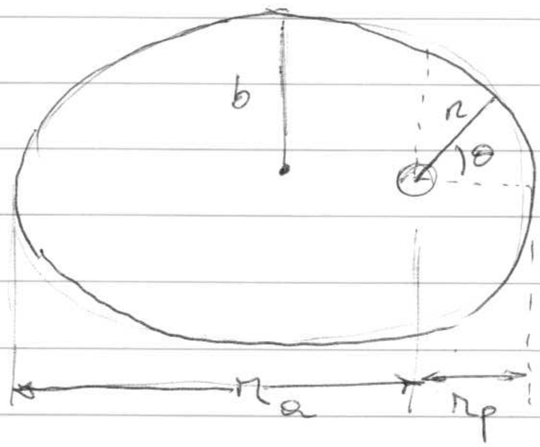
$$-\frac{\dot{r}}{r^2} = \frac{e}{r_c}\dot{\theta}\sin\theta = \frac{e}{r_c}\frac{h}{r^2}\sin\theta$$

$$\tau = \frac{2}{h}\pi a b = \frac{2\pi a r_c}{h\sqrt{1-e^2}}$$

thus

$$|\dot{r}|_{max} = \frac{e h}{r_c} = \frac{2\pi a e}{\tau\sqrt{1-e^2}}$$

(5)



With the center of the Earth as the origin, the equation for the orbit is

$$\frac{r_c}{r} = 1 + e\cos\theta$$

Also we know $r_{min} = a(1-e)$

$$r_{max} = a(1+e)$$

$r_{min} = r_p = 300 \text{ km} + r_\oplus = 6.67 \times 10^6 \text{ m}$

$r_{max} = r_a = 3500 \text{ km} + r_\oplus = 9.87 \times 10^6 \text{ m}$

$a = \frac{1}{2}(r_a + r_p) = 8.27 \times 10^6 \text{ m}$

Substituting into (2) gives $e = 0.193$

When $\theta = \pi$, $\frac{r_c}{r_{min}} = 1 + e$

which gives $r_c = 7.96 \times 10^6 \text{ m}$.

So the equation for the orbit is

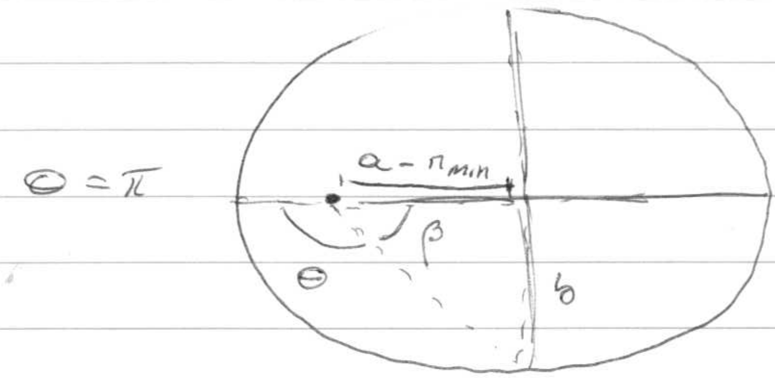
$$\frac{7.96 \times 10^6 \text{ m}}{r} = 1 - 0.193 \cos\theta$$

When $\theta = 3/2\,\pi$,

$$r = r_c = 7.96 \times 10^6 \text{ m}$$

The satellite is 1590 km above the Earth

(b)



$$\theta = 2\pi - \beta = 2\pi - \arctan \frac{b}{a - r_{min}}$$

Using $b = \sqrt{r_c a}$

$$\theta = 2\pi - \arctan \frac{\sqrt{r_c a}}{a - r_{min}} \approx 281^{\circ}$$

Substituting into (1) gives

$$r = 8.27 \times 10^{6} \text{ m}$$

which is 1900 km above the Earth.

(e) Let us obtain the major axis by exploiting its relationship to the total energy. In the following, let $M_\oplus$ be the mass of the Earth and m the mass of the satellite

$$E = -\frac{G M_\oplus m}{2a} = \frac{1}{2} m v_p^2 - \frac{G M_\oplus m}{r_p}$$

where r_p and v_p are the radius and velocity of the satellite's orbit at perigee. We can solve for a and use it to determine the radius of apogee by

$$r_a = 2a - r_p = r_p \left[\frac{2 G M_\oplus}{r_p v_p^2} - 1 \right]^{-1}$$

Inserting the values

$G = 6.67 \times 10^{-11}\ N \cdot m^2 \cdot kg^{-2}$

$M_\oplus = 5.974 \times 10^{24}\ kg$

$r_p = 6.59 \times 10^6\ m$

$v_p = 7.797 \times 10^3\ m\ s^{-1}$

we obtain $r_a \simeq 1.01\, r_p = 6.658 \times 10^6\ m$

or 288 km above the Earth's surface.

We may get the speed at apogee

from the conservation of angular

momentum

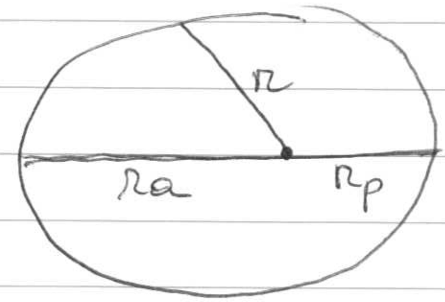
$$m\, r_a v_a = m\, r_p v_p$$

giving $v_a = 27.780$ km/h. The period

can be found from Kepler's 3rd law

$\tau^2 = \frac{4\pi a^3}{GM}$, substituting $\Rightarrow \tau = 1.49$ h

(7)



First, consider a kick Δv applied along the direction of travel at an arbitrary place in the orbit. We seek the optimum location to apply the kick

$\mathcal{E}_1$ = initial energy per unit mass

$$= \frac{1}{2} v^2 - \frac{GM}{r}$$

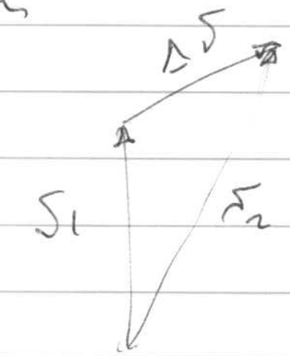
$\mathcal{E}_2$ = final energy per unit mass

$$= \frac{1}{2} (v + \Delta v)^2 - \frac{GM}{r}$$

We seek to maximise the energy gain $\varepsilon_2 - \varepsilon_1 = \frac{1}{2}\left(2v\Delta v + \Delta v^2\right)$

For a given Δv, this quantity is clearly maximum when v is maximum i.e at perigee.

Now consider a velocity kick Δv applied at perigee in an arbitrary direction



The final energy $\varepsilon_2 = \frac{1}{2} v_2^2 - \frac{GM}{r_p}$

This will be a maximum for a maximum $|v_2|$, which clearly occurs when v_1 and Δv are along the same direction

Thus, the most efficient way to change the energy of an elliptical orbit (for a single engine thrust) is by firing along the direction of travel at perigee

(8)

By conservation of angular momentum per unit mass h

$$r_a v_a = r_p v_p$$

or $v_a = \dfrac{r_p v_p}{r_a}$, substituting $v_a = 1608$ m/s

(8) Use conservation of energy for a spacecraft leaving the surface of the moon with just enough velocity v_{esc} to reach $r = \infty$

$$T_i + U_i = T_f + U_f$$

$$\frac{1}{2} m v_{esc}^2 - \frac{G M_{moon} m}{r_{moon}} = 0 + 0$$

where $M_{moon} = 7.36 \times 10^{22}$ kg

$r_{moon} = 1.74 \times 10^{6}$ m

Substituting gives $v_{esc} = 2380$ m/s

(10)

$$v_{max} = v + v_0$$

$$v_{min} = v - v_0$$

From conservation of angular momentum per unit mass we know $v_a r_a = v_b r_b$

or

$$v_{max} r_{min} = v_{min} r_{max}$$

$$\frac{r_{max}}{r_{min}} = \frac{v_{max}}{v_{min}} \quad (1)$$

Also we know

$$r_{min} = a(1-e) \quad (2)$$

$$r_{max} = a(1+e) \quad (3)$$

Dividing (3) by (2) and setting the result equal to (1) gives

$$\frac{r_{max}}{r_{min}} = \frac{1+e}{1-e} = \frac{v_{max}}{v_{min}}$$

$$v_{min}(1+e) = v_{max}(1-e)$$

$$e(v_{min} + v_{max}) = v_{max} - v_{min}$$

$$e(2v) = 2v_0$$

$$e = v_0/v$$