PROBLEMS # 6

① Inside the sphere the gravitational potential satisfies

$$\nabla^2 \phi = 4\pi G \rho(r) \qquad (1)$$

Since $\rho(r)$ is spherically symmetric ϕ is also spherically symmetric. Thus

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \phi}{\partial r}\right] = 4\pi G \rho(r) \qquad (2)$$

The field vector is independent from the radial distance. This fact implies

$$\frac{\partial \phi}{\partial r} = \text{constant} \equiv c \qquad (3)$$

Therefore, (2) becomes

$$\frac{2c}{r} = 4\pi G \rho \qquad (4)$$

or $$\rho = \frac{c}{2\pi G r} \qquad (5)$$

(2) In order to remove a particle from the surface of the Earth and transport it infinitely far away, the kinetic energy must equal the work required to move the particle from $r = R_\oplus$ to $r = \infty$ against the attractive gravitational force

$$\int_{R_\oplus}^{\infty} \frac{G M_\oplus m}{r^2}\, dr = \frac{1}{2} m v_0^2 \quad (1)$$

where $M_\oplus$ and $R_\oplus$ are the mass and the radius of the Earth, respectively and v_0 is the initial velocity of the particle at $r = R_\oplus$

Solving (1), we have the expression for

$$v_0 = \sqrt{\frac{2 G M_\oplus}{R_\oplus}} \quad (2)$$

Substituting $G = 6.67 \times 10^{-8}$ dyne-cm^2/g^2,

$M_\oplus = 5.98 \times 10^{27}$ g, $R_\oplus = 6.37 \times 10^{8}$ cm we have

$$v_0 \simeq 11.2 \text{ km/s} \quad (3)$$

(3) The potential energy corresponding to the force is

$$U = -\int f\,dx = mk^2 \int \frac{dx}{x^3} = -\frac{mk^2}{2x^2} \quad (1)$$

The central force is conservative and so the total energy is constant and equal to the potential energy at the initial position, $x = d$:

$$E = \text{constant} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \frac{k}{x^2} = -\frac{1}{2}\frac{mk}{d^2} \quad (2)$$

Rewriting this equation in an integrable form

$$dt = -\int_d^0 \frac{dx}{\sqrt{k^2\left[1/x^2 - 1/d^2\right]}} = -\frac{d}{k}\int_d^0 \frac{x\,dx}{\sqrt{d^2-x^2}} \quad (3)$$

where the choice of the negative sign for the radical insures that x decreases as t increases

$$t = \frac{d}{k}\sqrt{d^2-x^2}\,\Big|_d^0 \quad \text{or} \quad t = \frac{d^2}{k} \quad (4)$$

(4) The equation of motion is

$$m\ddot{x} = -\frac{GMm}{x^2} \qquad (1)$$

Using conservation of energy, we find

$$\frac{1}{2}\dot{x}^2 - GM\frac{1}{x} = E = -GM\frac{1}{x_\infty} \qquad (2)$$

$$\frac{dx}{dt} = -\sqrt{2GM\left[\frac{1}{x} - \frac{1}{x_\infty}\right]} \qquad (3)$$

where x_∞ is some fixed large distance.

Therefore, the time for the particle to travel from x_∞ to x is

$$t = -\int_{x_\infty}^{x} \frac{dx}{\sqrt{2GM\left[1/x - 1/x_\infty\right]}} = \frac{-1}{\sqrt{GM}}\int_{x_\infty}^{x}\sqrt{\frac{x x_\infty}{2(x_\infty - x)}}\,dx$$

Making the change of variable, $x \to y^2$

we obtain (after integration)

$$t = \sqrt{\frac{x_\alpha}{2GM}} \left[\sqrt{x(x_\alpha - x)} - x_\alpha \arcsin \sqrt{\frac{x}{x_\alpha}} \right]_{x_\alpha}^{x} \quad (4)$$

If one set $x = 0$ and $x = x_\alpha/2$ in (4), we can obtain the time for the particle to travel the total distance and the first half of the distance.

$$T_0 = \int_{x_\alpha}^{0} dt = \frac{\pi}{\sqrt{GM}} \left[\frac{x_\alpha}{2} \right]^{3/2} \quad (5)$$

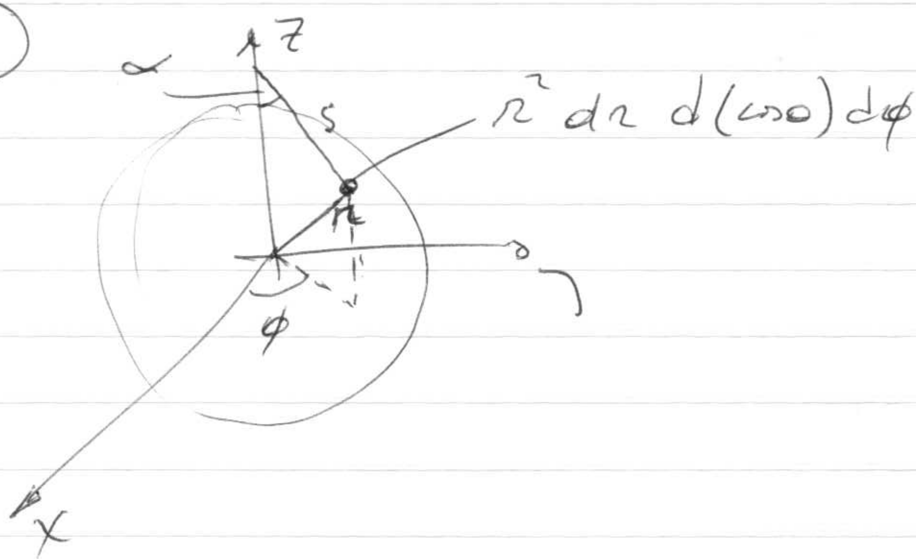
$$T_{1/2} = \int_{x_\alpha}^{x_\alpha/2} dt = \frac{1}{\sqrt{GM}} \left[\frac{x_\alpha}{2} \right]^{3/2} \left[1 + \frac{\pi}{2} \right] \quad (6)$$

Then $$\frac{T_{1/2}}{T_0} = \frac{1 + \pi/2}{\pi} \quad (7)$$

Evaluating this expresion $T_{1/2}/T_0 = 0.818$

or $\frac{T_{1/2}}{T_0} \sim \frac{9}{11}$

(5)



Since the problem has symmetry around the z axis, the force at the point P has only a z component. The contribution to the force from a small volume element is

$$dg_z = -G\frac{\rho}{s^2}\,r^2\,dr\,d(\cos\theta)\,d\phi\,\cos\alpha \quad (1)$$

where ρ is ~~the~~ ~~the~~ density.

Using

$$\cos\alpha = \frac{z - r\cos\theta}{s}$$

and integrating over the entire sphere we have

$$g_z = -G\rho \int_0^a r^2\,dr \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi \, \frac{z - r\cos\theta}{\left(r^2 + z^2 - 2rz\cos\theta\right)^{3/2}} \qquad (2)$$

Now we can obtain the integral of $\cos\theta$ as follows

$$I = \int_{-1}^{1} \frac{z - r\cos\theta}{\left(r^2 + z^2 - 2rz\cos\theta\right)^{3/2}} \, d(\cos\theta)$$

$$= -\frac{\partial}{\partial z} \int_{-1}^{1} \left(r^2 + z^2 - 2rz\cos\theta\right)^{-1/2} d(\cos\theta) \qquad (3)$$

$$I = -\frac{\partial}{\partial z}\left[-\frac{1}{2z}\left(r^2 + z^2 - 2rz\cos\theta\right)^{1/2}\right]_{-1}^{+1}$$

$$= -\frac{\partial}{\partial z}\left(\frac{2}{z}\right) = \frac{2}{z^2} \qquad (4)$$

Therefore, substituting (3) into (2) and performing the integral with respect to r and ϕ, we have

$$g_z = -G\rho\,\frac{a^3}{3}\,\frac{2}{z^2}\,2\pi$$

$$= -G\,\frac{4\pi}{3}\,a^3\,\rho\,\frac{1}{z^2} \qquad (5)$$

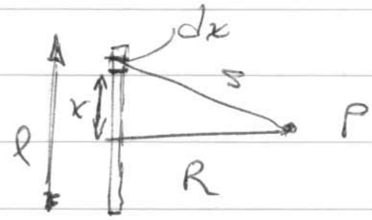
But $\frac{4}{3}\pi a^3\rho$ is equal to the mass of the sphere. Thus

$$g_z = -GM\,\frac{1}{z^2} \qquad (6)$$

Thus, as we expect, the force is the

same as a point mass M located at the center of the sphere.

(e)



The contribution to the potential at P from a small line element is

$$d\phi = -\frac{G \rho_\ell \, dx}{s}$$

where $\rho_\ell = M/\ell$ is the linear mass density. Integrating over the whole rod, we find the potential

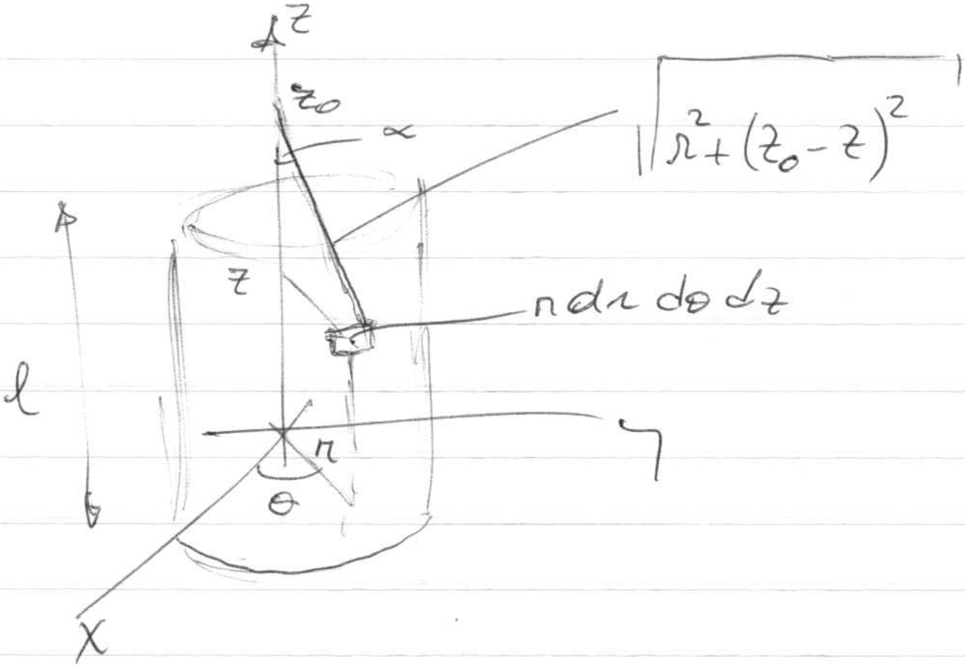
$$\phi = -G\frac{M}{\ell}\int_{-\ell/2}^{\ell/2} \frac{1}{\sqrt{x^2+R^2}}\,dx$$

$$\phi = -G\frac{M}{\ell}\ln\left[x+\sqrt{x^2+R^2}\right]_{-\ell/2}^{\ell/2} = -\frac{GM}{\ell}\ln\left[\frac{\frac{\ell}{2}+\sqrt{\frac{\ell^2}{4}+R^2}}{-\frac{\ell}{2}+\sqrt{\frac{\ell^2}{4}+R^2}}\right]$$

so that

$$\phi = -\frac{GM}{l} \ln\left[\frac{\sqrt{l^2+4R^2}+l}{\sqrt{l^2+4R^2}-l}\right]$$

(7)



Since the system is symmetric about the z-axis, the x and y components of the force vanish and we need to consider only the z-component of the force. The contribution to the force from a small element of volume at the point (r, θ, z) for a unit mass at $(0, 0, z_0)$ is

$$df_z = -G\rho \frac{r\,dr\,d\theta\,dz}{r^2+(z_0-z)^2} \cos\alpha$$

$$= -G\rho \frac{(z_0 - z)\, r\, dr\, \cancel{d\theta}\, dz}{\left[r^2 + (z_0 - z)^2\right]^{3/2}}$$

where ρ is the density of the cylinder and we have used

$$\cos\alpha = \frac{(z_0 - z)}{\sqrt{r^2 + (z_0 - z)^2}}$$

We can find the net gravitational force by integrating (1) over the entire volume of the cylinder we find

$$f_z = -G\rho \int_0^a r\,dr \int_0^{2\pi} d\theta \int_0^{\ell} dz \, \frac{z_0 - z}{\left[r^2 + (z_0 - z)^2\right]^{3/2}}$$

Changing the variable $x = z_0 - z$ we have

$$f_z = 2\pi G\rho \int_0^a r\,dr \int_{z_0}^{z_0 - \ell} \frac{x\,dx}{\left[r^2 + x^2\right]^{3/2}}$$

Using the standard integral

$$\int \frac{x\,dx}{\sqrt{(a^2 \pm x^2)^3}} = -\frac{1}{\sqrt{a^2 \pm x^2}}$$

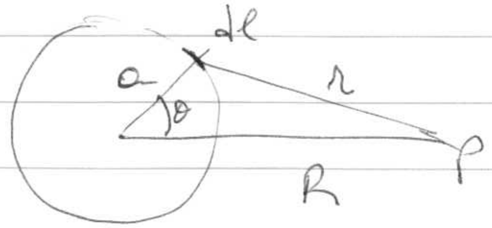
we obtain

$$g_z = -2\pi G \rho \int_0^a dr \left[\frac{r}{\sqrt{r^2 + (z_0 - l)^2}} - \frac{r}{\sqrt{r^2 + z_0^2}} \right]$$

$\Rightarrow$ after integration

$$g_z = -2\pi G \rho \left[\sqrt{a^2 + (z_0 - l)^2} - \sqrt{a^2 + z_0^2} + l \right]$$

(8)



The contribution to the potential at the point P from a small line element dl is

$$\phi = -G \int \frac{\rho_\ell \, d\ell}{r} \qquad (1)$$

where ρ_ℓ is the linear mass density which is expressed as $\rho_\ell = \dfrac{M}{2\pi a}$

Using $r = \sqrt{R^2 + a^2 - 2aR\cos\theta}$ and $d\ell = a\,d\theta$ we can write (1) as

$$\phi = -\frac{GM}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{R^2 + a^2 - 2aR\cos\theta}}$$

This is a general expression for the potential.

If R is much greater than a, we can expand the integrand in (2) using the binomial expansion

$$\frac{1}{\sqrt{R^2 + a^2 - 2aR\cos\theta}} = \frac{1}{R}\left[1 - \left(2\frac{a}{R}\cos\theta - \frac{a^2}{R^2}\right)\right]^{-1/2}$$

$$= \left[1 + \frac{1}{2}\left(2\frac{a}{R}\cos\theta - \frac{a^2}{R^2}\right) + \frac{3}{8}\left(2\frac{a}{R}\cos\theta - \frac{a^2}{R^2}\right)^2\right]\frac{1}{R} + \ldots$$

(3)

if we neglect terms of order $\left(\frac{a}{R}\right)^3$ and higher in (3), the potential becomes

$$\phi \simeq -\frac{GM}{2\pi R}\int_0^{2\pi}\left[1 + \frac{a}{R}\cos\theta - \frac{a^2}{2R^2} + \frac{3}{2}\frac{a^2}{R^2}\cos\theta\right]d\theta$$

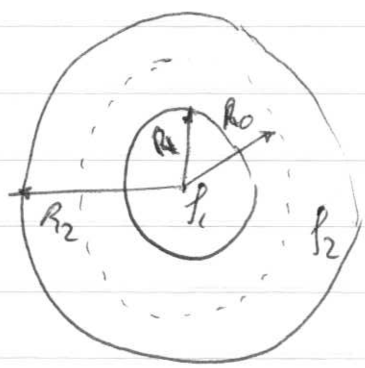
$$= -\frac{2GM}{2\pi R}\left[2\pi - \pi\frac{a^2}{R^2} + \frac{3}{2}\pi\frac{a^2}{R^2}\right]$$

or

$$\phi(R) = -\frac{GM}{R}\left[1 + \frac{1}{4}\frac{a^2}{R^2}\right]$$

(9)

we notice that the first term in (5) is the potential when mass M is concentrated in the center of the ring. Of course this is a very rough estimate and the first correction term is $-\frac{GMa^2}{4R^3}$



R_0 = position of particle.

For $R_1 < R_0 < R_2$ we calculate the force by assuming that all mass for which $r < R_0$ is at $r = 0$, and neglect mass for which $r > R_0$; the force is then radially inward direction $(-e_r)$

The magnitude of the force is

$$F = \frac{GMm}{R_0^2}$$

where M = mass for which $r < R_0$

$$M = \frac{4}{3} \pi R_1^3 \rho_1 + \frac{4}{3} \pi \left(R_0^3 - R_1^3 \right) \rho_2$$

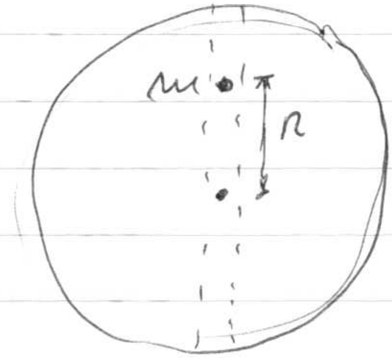
So $$F = -\frac{4\pi G m}{3 R_0^2} \left(\rho_1 R_1^3 + \rho_2 R_0^3 - \rho_2 R_1^3 \right) \vec{e}_r$$

$$F = -\frac{4}{3} \pi G m \left[\frac{(\rho_1 - \rho_2) R_1^3}{R_0^2} + \rho_2 R_0 \right] \vec{e}_r$$

(1c) When the mass is at a distance r from the center of the earth, the force in the inward radial direction and has magnitude

$$F_r = \frac{Gm}{r^2} \left[\frac{4}{3} \pi r^3 \rho \right]$$

where ρ is the mass density of the Earth.



The equation of motion is

$$F_r = m\ddot{r} = -\frac{Gm}{r^2}\left[\frac{4}{3}\pi r^3 \rho\right]$$

$$\ddot{r} + \omega^2 r = 0 \quad \text{where} \quad \omega^2 = \frac{4\pi G\rho}{3}$$

This is the equation for a simple harmonic oscillator. The period is

$$T = \frac{2\pi}{\omega} = \sqrt{\frac{3\pi}{G\rho}}$$

Substituting in values gives a period of about 84 minutes

(11)

The maximum height cause by the moon is

$$h_{moon} = \frac{3 G M_{moon} r^2}{2 g D^3}$$

and equivalently

$$h_{\odot} = \frac{3 G M_{\odot} r^2}{2 g R^3}$$

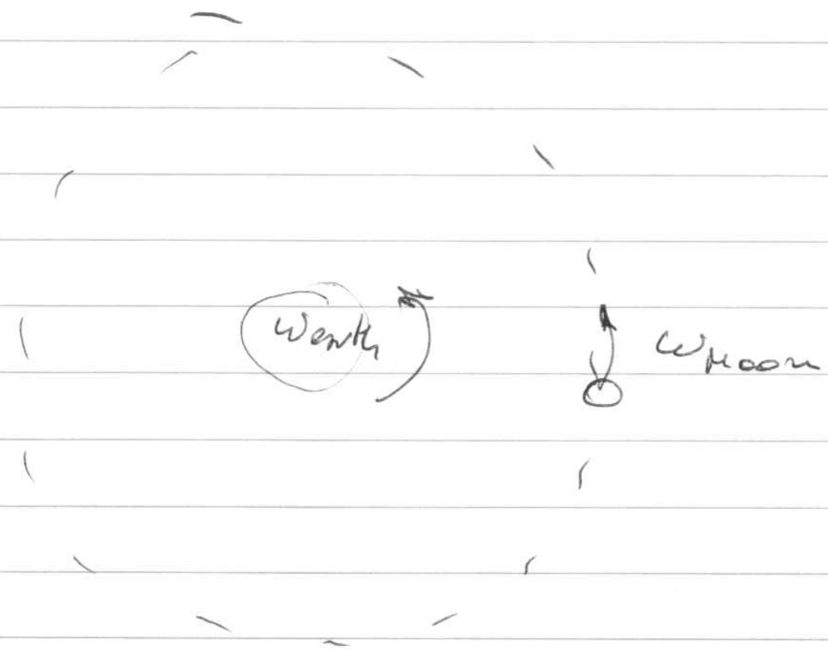
$$\Rightarrow \frac{h_{moon}}{h_{\odot}} = \frac{M_m}{M_{\odot}} \left[\frac{R}{D} \right]^3$$

Substitution of the known values gives

$$\frac{h_{moon}}{h_{\odot}} = \frac{7.350 \times 10^{22} \text{ kg}}{1.993 \times 10^{30} \text{ kg}} \left[\frac{1.495 \times 10^{11} \text{ m}}{3.84 \times 10^{8} \text{ m}} \right]^3$$

$$= 2.2$$

12



Because the moon orbit about the Earth is in the same sense as the Earth rotation, the difference of their frequencies will be half the observed frequencies at which we see high tides

Thus

$$\frac{1}{2T_{tides}} = \frac{1}{T_{earth}} - \frac{1}{T_{moon}}$$

$T_{tides} \simeq 12$ hours, 27 minutes

(13) For a spherical Earth, the difference in gravitational field strength between the poles and the equator is only the centripetal term

$$g_{poles} - g_{equator} = \Omega^2 R$$

For $\Omega = 7.3 \times 10^{-5}\ s^{-1}$ and $R = 6370$ km this difference is only 34 mm/s^2.
The disagreement with the exact result can be explained by the fact that the Earth is really an oblate spheroid, another consequence of rotation. To qualitatively describe this effect, approximate the real Earth as a somewhat smaller sphere with a massive belt around

the equator. It can be shown with a bit more of analysis that the belt pulls inward at the poles more than it does at the equator. The next order of correction is the quadrupoles correction to the gravitational potential of the Earth.