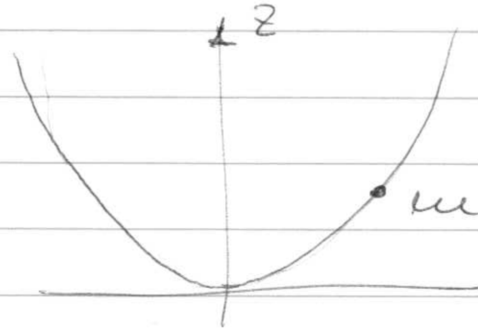
PROBLEMS # 5

(1)

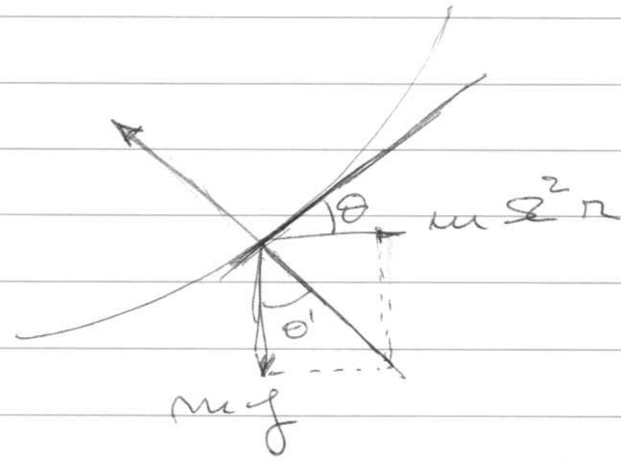


$z = f(r)$

Consider a small mass m on the surface of the water. In the rotating frame the mass is at rest. The nonfictitious forces are gravity and the force due to the pressure gradient, which is normal to the surface in equilibrium

$$m\vec{a}' = m\vec{g} + \vec{F}_n - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

where F_n is due to the pressure gradient. Since in the rotating system the mass is at rest $\dot{a}' = 0$



Since $\ddot{a}' = 0$, the sum of the gravitational and centrifugal forces must also be normal to the surface

Thus $\theta' = \theta$

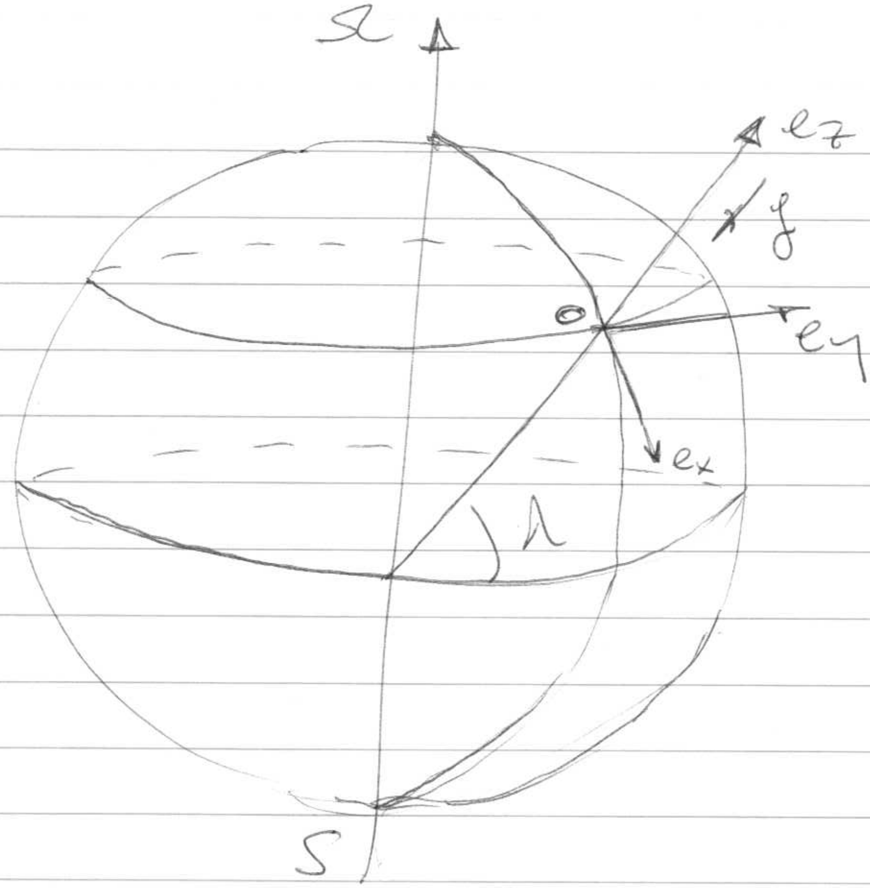
$$\tan\theta' = \tan\theta = \frac{\Omega^2 r}{g}$$

But $\tan\theta = \frac{dz}{dr} \Rightarrow \frac{dz}{dr} = \frac{\Omega^2}{g} r$

$$z = \frac{\Omega^2}{2g} r^2 + \text{constant}.$$

The shape is a circular paraboloid.

(2)



$$m\vec{a}' = m\vec{g} - 2\vec{\Omega} \times \vec{v}'$$

The prime system is a rotating frame fixed to Earth.

The acceleration due to gravity g is along the plumb line. We choose the z axis directed vertically outward (along $-g$) from the surface of the Earth. With this definition of $\bar{e}_z$, we complete the construction of a right-hand coordinate system by specifying $\vec{e}_x$ to be in the southerly direction and $\vec{e}_y$ in the easterly direction. We make the approximation that the distance of the fall is sufficiently small that g remains constant during the process. Because we have chosen the origin of the rotating system to be in the Northern hemisphere we have

$$\Omega_x = -\Omega \cos\lambda$$

$$\Omega_y = 0$$

$$\Omega_z = \Omega \sin\lambda$$

Although the Coriolis force produces small velocity components in the $\bar{e}_y$ and $\bar{e}_x$ directions, we can certainly neglect $\dot{x}$ and $\dot{y}$ components with $\dot{z}$, the vertical velocity. $\Rightarrow$ $\dot{x} \simeq 0$, $\dot{y} \simeq 0$, and $\dot{z} \simeq -gt$, where we obtain $\dot{z}$ by considering a fall from rest. Thus

$$\bar{\Omega} \times \bar{v}' \simeq \begin{vmatrix} e_x & e_y & e_z \\ -\Omega \cos\lambda & 0 & \Omega \sin\lambda \\ 0 & 0 & -gt \end{vmatrix}$$

$$= -(\Omega g t \cos\lambda)\, \bar{e}_y$$

The components of g are $g_x = 0$, $g_y = 0$, $g_z = -g$

So the equation of motion

(neglecting terms in Ω^2) become

$$(a')_x = \ddot{x} \simeq 0$$

$$(a')_y = \ddot{y} \simeq 2\Omega g t \cos\lambda$$

$$(a')_z = \ddot{z} \simeq -g$$

Thus, the effect of the Coriolis force is to produce an acceleration in the $\hat{e}_y$, or easterly direction. Integration of the equations of motion leads to

$$y(t) = \frac{1}{3} \Omega g t^3 \cos\lambda$$

where $y = 0$ and $\dot{y} = 0$ at $t = 0$.

The integration of $\ddot{z}$ yields the particular result for the distance of fall

$$z(t) \simeq z(0) - \frac{1}{2} g t^2$$

and the time of fall from $h = z(0)$ is given by $t = \sqrt{2h/g}$

$\Rightarrow$ the deflection d of the particle dropped from rest at a height h at northern latitude λ is

$$d \simeq \frac{1}{3} \omega \cos\lambda \sqrt{\frac{8h^3}{g}}$$

(3) Choose the coordinate x, y, z as in the previous problem. Then the velocity of the particle and the rotation frequency of the Earth are expressed by

$$\left\{\begin{array}{l} V = (0, 0, \dot{z}) \\ \Omega = (-\Omega\cos\lambda, 0, \Omega\sin\lambda) \end{array}\right. \qquad (1)$$

so that the acceleration due to the Coriolis force is

$$\vec{a} = -2\vec{\Omega}\times\dot{\vec{r}} = 2\Omega(0, -\dot{z}\cos\lambda, 0) \qquad (2)$$

This acceleration is directed along the y axis. Hence as the particle moves along the z axis, it will be accelerated along the y axis:

$$\ddot{y} = -2\Omega\dot{z}\cos\lambda \qquad (3)$$

Now, the equation of motion for the particle along the z axis is

$$\dot{z} = v_0 - gt \qquad (4)$$

$$z = v_0 t - \frac{1}{2} g t^2 \qquad (5)$$

where v_0 is the initial velocity and is equal to $v_0 = \sqrt{2gh}$ if the highest point the particle can reach is h.

From (3) we have

$$\dot{y} = -2\Omega z \cos\lambda + C \qquad (6)$$

~~The~~ But the initial condition $\dot{y}(z=0) = 0$ implies $C = 0$. Substituting (5) into (6) we find

$$\dot{y} = -2\Omega \cos\lambda \left(v_0 t - \frac{1}{2} g t^2\right)$$

$$= \Omega \cos\lambda \left(g t^2 - 2 v_0 t\right) \qquad (7)$$

Integrating and using the initial condition $y(t=0)=0$, we find

$$y = \Omega \cos\lambda \left[\frac{1}{3} g t^3 - v_0 t^2 \right] \qquad (8)$$

From (5) the time the particle strikes the ground ($z=0$) is

$$0 = \left(v_0 - \frac{1}{2} g t\right) t$$

so that

$$t = \frac{2 v_0}{g}$$

Substituting this value into (8)

$$y = \Omega \cos\lambda \left[\frac{1}{3} g \, \frac{8 v_0^3}{g^3} - v_0 \frac{4 v_0^2}{g} \right]$$

$$= -\frac{4}{3} \Omega \cos\lambda \frac{v_0^3}{g^2} \qquad (10)$$

If we use $v_0 = \sqrt{2gh}$ and (10) we find

$$y = -\frac{4}{3}\,\Omega \cos\lambda \sqrt{\frac{8h^3}{g}}$$

The negative sign of the displacement shows that the particle is displaced to the west.

(4) Choosing the same coordinate system than in problems (2) and (3) we see that the lateral deflection of the projectile is in the x direction and that the acceleration is

$$a_x = \ddot{x} = 2\,\Omega_z v_y = 2(\Omega \sin\lambda)(v_0 \cos\alpha)$$

Integrating this expression twice and using initial conditions, $\dot{x}(0)=0$ and $x(0)=0$, we obtain

$$x(t) = \Omega v_0 t^2 \cos\alpha \sin\lambda$$

Now we treat the z motion of the projectile as if it were undisturbed by the Coriolis force. In this approximation we have

$$z(t) = v_0 t \sin\alpha - \frac{1}{2} g t^2$$

from which the time T of impact is obtained by setting $z=0$

$$T = \frac{2 v_0 \sin\alpha}{g}$$

Substituting this value of T we

find the lateral deflection at impact

to be

$$x(T) = \frac{4 \Omega v_0^3}{g^2} \sin\lambda \cos\alpha \sin^2\alpha$$

(5)

In the previous problem we assumed the z motion to be unaffected by the Coriolis force. Actually, of course, there is an upward acceleration given by $-2\Omega_x v_y$ so that

$$\ddot{z} = 2\Omega v_0 \cos\alpha \cos\lambda - g \qquad (1)$$

from which the time of flight is obtained by integrating twice, using initial conditions, and then setting $z = 0$:

$$T' = \frac{2 v_0 \sin\alpha}{g - 2\Omega v_0 \cos\alpha \cos\lambda} \qquad (2)$$

Now the acceleration in the y

direction is

$$a_y = \ddot{y} = 2\Omega_x v_z$$

$$= 2(-\Omega\cos\lambda)(v_0\sin\alpha - gt) \quad (3)$$

Integrating twice and using initial

conditions $\dot{y}(0) = v_0\cos\alpha$ and $y(0) = 0$

we obtain

$$y(t) = \frac{1}{3}\Omega g t^3\cos\lambda - \Omega v_0 t^2\cos\lambda\sin\alpha + v_0 t\cos\alpha \quad (4)$$

Substituting (2) into (4), the range R

is

$$R = \frac{8}{3}\frac{\Omega v_0^3 g\sin^3\alpha\cos\lambda}{(g - 2\Omega v_0\cos\alpha\cos\lambda)^3} - \frac{4\Omega v_0^3\sin^3\alpha\cos\lambda}{(g - 2\Omega v_0\cos\alpha\cos\lambda)^2}$$

$$+ \frac{2v_0^2\cos\alpha\sin\alpha}{g - 2\Omega v_0\cos\alpha\cos\lambda}$$

We now expand each of these three terms, retaining quantities up to order Ω. In the first two terms this amounts to neglecting $2\Omega v_0 \cos\alpha \cos\lambda$ compared to g in the denominator. But in the 3rd term we must use

$$\frac{2 v_0^2 \cos\alpha \sin\alpha}{g\left[1 - \frac{2\Omega v_0}{g}\cos\alpha\cos\lambda\right]} =$$

$$\simeq \frac{2 v_0^2}{g} \cos\alpha \sin\alpha \left[1 + \frac{2\Omega v_0}{g} \cos\alpha \cos\lambda\right]$$

$$= R_0 + \frac{4\Omega v_0^3}{g^2} \sin\alpha \cos^2\alpha \cos\lambda$$

where R_0 is the range when Coriolis effects are neglected

$$R_0 = \frac{2 v_0^2}{g} \cos\alpha \sin\alpha$$

The range difference now becomes

$$\Delta R = \frac{4 \Omega v_0^3}{g^2} \cos\lambda \left(\sin\alpha \cos^2\alpha - \frac{1}{3} \sin^3\alpha \right)$$

Substituting for v_0 in terms of R_0

we finally have

$$\Delta R = \sqrt{\frac{2 R_0}{g}}\, \Omega \cos\lambda \left(\cot^{1/2}\alpha - \frac{1}{3} \tan^{3/2}\alpha \right)$$