PROBLEMS # 4

① The position and velocity for a simple harmonic oscillator are given by

$$x = A \sin \omega_0 t \qquad (1)$$
$$\dot{x} = \omega_0 A \cos \omega_0 t \qquad (2)$$

where $\omega_0 = \sqrt{K/m}$

The time average of the kinetic energy is

$$\langle T \rangle = \frac{1}{\tau} \int_t^{t+\tau} \frac{1}{2} m \dot{x}^2 \, dt \qquad (3)$$

where $\tau = \frac{2\pi}{\omega_e}$ is the period of oscillation

$$\langle T \rangle = \frac{1}{2\tau} m A^2 \omega_0^2 \int_t^{t+\tau} \cos^2 \omega_0 t \, dt \quad (4)$$

$$\langle T \rangle = \frac{m A^2 \omega_0^2}{4} \qquad (5)$$

In the same way, the time average of potential energy is

$$\langle U \rangle = \frac{1}{\tau}\int_t^{t+\tau} \frac{1}{2} k x^2 \, dt$$

$$= \frac{1}{2\tau} k A^2 \int_t^{t+\tau} \sin^2 \omega_0 t \, dt$$

$$= \frac{kA^2}{4} \qquad (6)$$

since $\omega_0 = k/m \Rightarrow$

$$\langle U \rangle = \frac{mA^2\omega_0^2}{4} \qquad (7)$$

Then we see that $\langle U \rangle = \langle T \rangle$ (8)

This result is reasonable to expect from the conservation of the total energy

$$E = T + U \qquad (9)$$

This equality is valid instantaneously, as well as in average. On the other hand, when T and U are expressed by (1) and (2), we note that they are described by exactly the same function displaced by a time $\tau/2$

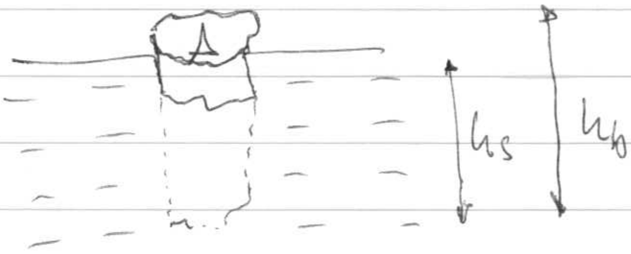
$$T = \frac{mA^2\omega_0^2}{2} \cos^2 \omega_0 t \qquad (10)$$

$$U = \frac{mA^2\omega_0^2}{2} \sin^2 \omega_0 t \qquad (11)$$

Therefore the time average of T and U must be equal. By taking the time average of (9) we find

$$\langle T \rangle = \langle U \rangle = E/2 \qquad (12)$$

(2)



Let A be the cross sectional area of the floating body, h_b is its height, and h_s the height of its submerged part, and let ρ and ρ_0 denote the mass densities of the body and the fluid respectively.

The volume of the displaced fluid is therefore $V = A h_s$. The mass of the body is $M = \rho A h_b$.

There are two forces acting on the body: that due to gravity (Mg) and that due to the fluid pushing the body up.

$$-\rho_0 g V = -\rho_0 g h_s A.$$

The equilibrium situation occurs when the total force vanishes

$$0 = Mg - \rho_0 g V$$

$$= \rho g A h_b - \rho_0 g h_s A \qquad (1)$$

which gives the relation between

$$h_s = h_b \frac{\rho}{\rho_0} \qquad (2)$$

For small displacement about the equilibrium position

$$h_s \longrightarrow h_s + x$$

and so (1) becomes

$$M\ddot{x} = \rho A h_b \ddot{x} = \rho g A h_b - \rho_0 g (h_s + x) A \qquad (3)$$

Using (2) we have

$$\rho A h_b \ddot{x} = -\rho_0 g x A$$

or

$$\ddot{x} + g \frac{\rho_0}{\rho h_b} x = 0$$

Thus, the motion is oscillatory with angular frequency

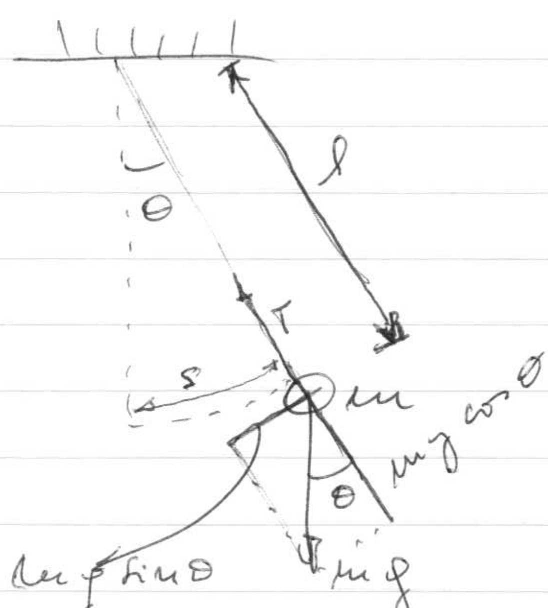
$$\omega^2 = g \frac{\rho_0}{\rho h_0} = \frac{g}{h_s} = \frac{gA}{V}$$

The period of oscillations is therefore

$$\tau = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{V}{gA}}$$

Substituting the given value $\tau = 0.18\,s$

(3)



Using tangential components Newton's second law gives

$$-mg\sin\theta = m\frac{d^2s}{dt^2}$$

where the arc length s is related to the angle θ by $s = l\theta$. Repeatedly differentiating both sides of $s = l\theta$ leads to $\frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}$

The equation of motion is then

$$-ml\ddot{\theta} = mg\sin\theta$$

If θ is sufficiently small we can approximate $\sin\theta \simeq \theta$ and then

$$\ddot{\theta} = -\frac{g}{l}\theta$$

which has the oscillatory solution

$$\theta(t) = \theta_0 \cos w_0 t$$

where $w_0 = \sqrt{g/l}$ and where θ_0 is the amplitude. If there is a retarding force $2m\sqrt{gl}\,\dot\theta$, the eq of motion becomes

$$-ml\ddot\theta = mg\sin\theta + 2m\sqrt{gl}\,\dot\theta$$

or setting $\sin\theta \simeq \theta$ and rewriting

$$\ddot\theta + 2\omega_0\dot\theta + \omega_0^2\theta = 0$$

Comparing this equation with the standard equation for damped motion

$$\ddot x + 2\beta\dot x + \omega_0^2 x = 0$$

we identify $\omega_0 = \beta$. This is the case for critical damping, so the solution for $\theta(t)$ is

$$\theta(t) = (A+Bt)\,e^{-\omega_0 t}$$

For initial conditions $\theta(0) = \theta_0$ and

$\dot{\theta}(0) = 0$ we find

$$\theta(t) = \theta_0 \left(1 + \omega_0 t\right) e^{-\omega_0 t}$$

(4) The equation of motion for

$$0 \leq t \leq t_0 \text{ is}$$

$$m\ddot{x} = -k\,(x - x_0) + F = -kx + (F + k x_0) \quad (1)$$

while for $t \geq t_0$, the equation is

$$m\ddot{x} = -k\,(x - x_0) = -kx + k x_0 \quad (2)$$

It is convenient to define

$$\xi = x - x_0$$

which transform (1) and (2) into.

$$m\ddot{\xi} = -k\xi + F \;; \qquad 0 \leq t \leq t_0 \quad (3)$$

$$m\ddot{\xi} = -k\xi \;; \qquad t > t_0 \quad (4)$$

The homogeneous solutions for both (3) and (4) are of familiar form

$$\xi(t) = A e^{i\omega t} + B e^{-i\omega t} \text{, where}$$

$\omega = \sqrt{k/m}$. A particular solution for (3) is $\xi = F/k$

Then the general solutions for (3) and (4) are

$$\xi_- = \frac{F}{k} + A e^{i\omega t} + B e^{-i\omega t} \; ; \quad 0 \le t \le t_0 \qquad (5)$$

$$\xi_+ = C e^{i\omega t} + D e^{-i\omega t} \; ; \quad t \geq t_0 \qquad (6)$$

To determine the constants we use the initial conditions

$x(t=0) = x_0$ and $x(t=0) = 0$. Thus;

$$\xi_-(t=0) = \dot{\xi}_-(t=0) = 0 \qquad (7)$$

The conditions give two equations for A and B

$$0 = \frac{F}{k} + A + B$$

$$0 = i\omega (A - B)$$

Then $A = B = -\dfrac{F}{2k}$ (8)

and from (5) we have

$$\xi_- = x - x_0 = \frac{F}{k}(1 - \cos\omega t) \; ; \; 0 \le t \le t_0 \qquad (9)$$

Since for any physical motion, x and $\dot{x}$ must be continuous, the values of $\xi_-(t = t_0)$ and $\dot{\xi}_-(t = t_0)$ are the initial conditions for $\xi_+(t)$ which are needed to determine C and D

$$\xi_+(t=t_0) = \frac{F}{k}\left(1-\cos\omega t_0\right) = C e^{i\omega t_0} + D e^{-i\omega t_0}$$

$$\dot{\xi}_+(t=t_0) = \frac{F}{k}\,\omega \sin\omega t_0 = i\omega\left[C e^{i\omega t_0} - D e^{-i\omega t_0}\right]$$

These equations can be re-written as

$$C e^{i\omega t_0} + D e^{-i\omega t_0} = \frac{F}{k}\left(1-\cos\omega t_0\right)$$

$$C e^{i\omega t_0} - D e^{-i\omega t_0} = -\frac{iF}{k}\sin\omega t_0$$

Then by adding and subtracting one from another one obtain:

$$C = \frac{F}{2K}\, e^{-i\omega t_0}\left(1-e^{i\omega t_0}\right)$$

$$D = \frac{F}{2K}\, e^{i\omega t_0}\left(1-e^{-i\omega t_0}\right)$$

Substitution into (6) leads to

$$\xi_{+} = \frac{f}{2k}\left[\left(e^{-i\omega t_0} - 1\right) e^{i\omega t} + \left(e^{i\omega t_0} - 1\right) e^{-i\omega t}\right]$$

$$= \frac{f}{2k}\left[\left(e^{i\omega(t-t_0)} - e^{i\omega t} + e^{-i\omega(t-t_0)} - e^{-i\omega t}\right.\right.$$

$$= \frac{f}{k}\left[\cos\omega(t-t_0) - \cos\omega t\right]$$

Thus

$$x - x_0 = \frac{f}{k}\left[\cos\omega(t-t_0) - \cos\omega t\right]; \ t > t_0$$

(5) The amplitude of a damped oscillator is expressed by

$$x(t) = A e^{-\beta t} \cos(\omega_n t + \delta)$$

Since the amplitude decreases to $1/e$ after n periods, we have

$$\beta n T = \beta n \frac{2\pi}{\omega_n} = 1$$

Substituting this relation into the equation connecting ω_r and ω_0 (the frequency of undamped oscillations)

$\omega_r^2 = \omega_0^2 - \beta$ we have

$$\omega_0^2 = \omega_r^2 + \left[\frac{\omega_r}{2\pi n}\right]^2 = \omega_r^2\left[1 + \frac{1}{4\pi^2 n^2}\right]$$

$\Rightarrow$

$$\frac{\omega_r}{\omega_0} = \sqrt{1 + \frac{1}{4\pi^2 n^2}}$$

so that

$$\frac{\omega_r}{\omega_0} \simeq 1 - \frac{1}{8\pi^2 n^2}$$

(6) The amplitude of a forced oscillator is

$$D = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}$$

At the resonance frequency,

$\omega = \omega_R = \sqrt{\omega_0^2 - 2\beta^2}$, D becomes

$$D_R = \frac{A}{2\beta\sqrt{\omega_0^2 - \beta^2}}$$

Let us find the frequency $\omega = \omega'$, at which the amplitude is $\frac{1}{\sqrt{2}} D_R$

$$\frac{1}{\sqrt{2}} D_R = \frac{1}{\sqrt{2}} \frac{A}{2\beta\sqrt{\omega_0^2 - \beta^2}} = \frac{A}{\sqrt{(\omega_0^2 - \omega'^2)^2 + 4\omega'^2\beta^2}}$$

Solving this equation for ω', we find

$$\omega'^2 = \omega_0^2 - 2\beta^2 \mp 2\beta\omega_0 \left[1 - \frac{\beta^2}{\omega_0^2}\right]^{1/2}$$

For a lightly damped oscillator, β is small and the terms in β^2 can be neglected. Therefore

$$\omega'^2 = \omega_0^2 \pm 2\beta\omega_0$$

or

$$\omega' = \omega_0 \left[1 \pm \frac{\beta}{\omega_0}\right]$$

which gives

$$\Delta\omega = (\omega_0 + \beta) - (\omega_0 - \beta) = 2\beta$$

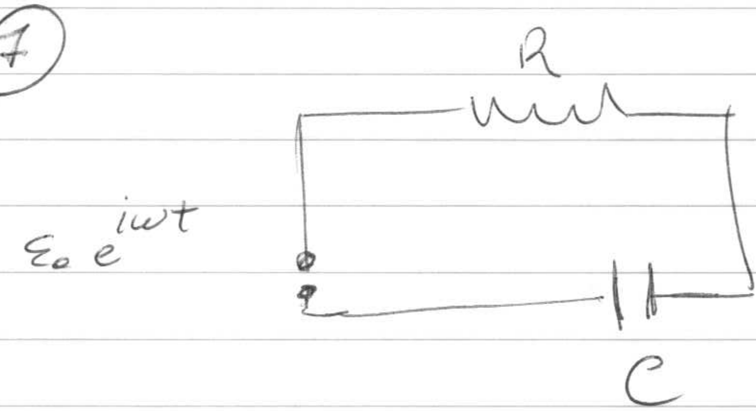
We also can approximate ω_R for a lightly damped oscillator

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2} \simeq \omega_0$$

Therefore, Q for a lightly damped oscillator becomes

$$Q \simeq \frac{\omega_0}{2\beta} \simeq \frac{\omega_0}{\Delta\omega}$$

(7)



The circuit equation is

$$R\dot{q} + \frac{q}{C} = \mathcal{E}_0 e^{i\omega t} \qquad (1)$$

Differentiating with respect to time eq (1) can be written as

$$R\dot{I} + \frac{1}{C} I = i\omega \mathcal{E}_0 e^{i\omega t} \qquad (2)$$

we try a steady state solution for

$I(t)$ of the form

$$I(t) = I_0 \, e^{i\omega t} \qquad (3)$$

where I_0 may be complex. Substituting

(3) into (2) and dividing by $i\omega e^{i\omega t}$,

we obtain

$$I_0 = \frac{\varepsilon_0}{Z}$$

where $\quad Z = R - i \dfrac{1}{\omega C}$

writing I_0 in terms of a magnitude

and phase factor we have

$$I_0 = \frac{\varepsilon_0}{\left[R^2 + \dfrac{1}{\omega^2 C^2}\right]^{1/2}} \, e^{-i\phi}$$

where

$$\phi = \arctan\left[-1/\omega C R\right]$$

The current can now be expressed as

$$I(t) = \frac{\varepsilon_0 \omega C}{\left(\omega^2 C^2 R^2 + 1\right)} e^{i\left[\omega t - \arctan\left(-1/\omega C R\right)\right]} \quad (8)$$

Taking only real part of (8), the actual current is

$$I(t) = \frac{\varepsilon_0 \omega C}{\left(\omega^2 C^2 R^2 + 1\right)} \cos\left[\omega t - \arctan\left(\frac{-1}{\omega C R}\right)\right] \quad (9)$$

For $\omega \to 0$ we can expand the expression in (9) in a power series, and keeping only terms up to second order in ω, we have

$$\phi = \arctan\left[-1/\omega C R\right] \simeq -\pi/2 + \eta$$

where η is a very small quantity

From (10), for $\omega \to 0$

$$-\frac{1}{\omega CR} \simeq \tan\left[-\frac{\pi}{2}+\eta\right] = -\cot\eta \quad (11)$$

$$\omega CR \cong \tan\eta \simeq \eta \quad (12)$$

Then from (10) and (12)

$$\cos\left[\omega t - \arctan\left(\frac{-1}{\omega CR}\right)\right] = \cos\left[\omega\left(t - CR + \pi/2\right)\right]$$

$$= -\sin\left[\omega(t - CR)\right]$$

$$\simeq -\left[\omega(t - CR)\right]$$

Then the term $\dfrac{\omega}{(\omega^2C^2R^2+1)}$ to second

order in ω is $\dfrac{\omega}{(\omega^2C^2R^2+1)} \simeq \omega$

As the frequency of the alternating emf approaches zero, the current can be written to second order in ω, from (9), (13), and (14), as

$$I(t) \approx \mathcal{E}_0 C \omega^2 (RC - t) \text{ for } \omega \to 0$$