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| **Problems set # 9** | **Physics 303** | November 4, 2014 |
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1. Three distinguishable particles can occupy three states. Find all macrostates and the numbers of microstates realizing each macrostate. Compile your results into a table. Which macrostate has the highest statistical weight? What is the total number of microstates that can be found immediately? Is the sum of the microstates realizing each mucrostate equal to this expected total number?

Solution: The results are given in the table below. The total number of microstaes is $3^3 = 27$.

2. Use the method of Lagrange multipliers to find the area of the largest rectangle that can be inscribed into the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: Minimize the function $\Phi(x,y) = xy - \lambda(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1)$, with Lagrange multiplier $\lambda$. The expressions for the extrema are: $0 = \frac{\partial\Phi}{\partial x} = y - 2\lambda x/a^2$, $0 = \frac{\partial\Phi}{\partial y} = x - 2\lambda y/b^2$, $0 = \frac{\partial\Phi}{\partial\lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$. Expressing $y = 2\lambda x/a^2$ from $\partial\Phi/\partial x$ and inserting the result into the $\partial\Phi/\partial y$ equation you obtain $0 = x - 4\lambda^2 x/(ab)^2$, that is $\lambda = ab/2$. Then, it follows that $y = x\frac{b}{a}$. Inserting this into the third extrema equation you obtain $0 = \frac{x^2}{a^2} + \frac{x^2}{a^2} - 1$ and so $x = a/\sqrt{2}$, $y = b/\sqrt{2}$. The maximal area is now $A_{\rm max} = xy = ab/2$. A *posteriori* you can see that with the smart choice of variables $\bar{x} \equiv x/a$ and $\bar{y} \equiv y/b$ you can achieve a more elegant and probably completely symmetric solution of the equations.

3. *(i)* Find the density of states for quantum particles in a one-dimensional rigid box. *(ii)* Generalize the result for two and three dimensions.

Solution: *(i)* In one dimension the energy levels are given by $\varepsilon_n = \frac{\hbar^2 k_\nu^2}{2m}$, with $k_\nu = \nu\frac{\pi}{L}$, and $\nu = 1, 2, 3, \cdots$. To calculate the density of states defined by $dn_\varepsilon = \rho(\varepsilon)d\varepsilon$, start with $dn_\nu = d\nu$, as the number of states (enegy levels) in the interval $d\nu$ of the quantum number $\nu$. Using the expression for the quantized wave vector, write $dn_\nu = d\nu$ in terms of $k$ as $dn_k = \frac{L}{\pi}\, dk$. Using the relation between the wave vector and the energy, $k = \sqrt{2m\varepsilon}/\hbar$, you obtain $dn_\varepsilon = \frac{L}{\pi}\frac{dk}{d\varepsilon}d\varepsilon = \frac{L}{2\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2}\frac{d\varepsilon}{\sqrt{\varepsilon}}$, thus

| Macrostates | | | Numbers of microstates |
|---|---|---|---|
| 3 | 0 | 0 | 1 |
| 0 | 3 | 0 | 1 |
| 0 | 0 | 3 | 1 |
| 2 | 1 | 0 | 3 |
| 2 | 0 | 1 | 3 |
| 1 | 2 | 0 | 3 |
| 1 | 0 | 2 | 3 |
| 0 | 2 | 1 | 3 |
| 0 | 1 | 2 | 3 |
| 1 | 1 | 1 | 6 |

the density of states is given by $\rho(\varepsilon) = \frac{L}{2\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2}\frac{1}{\sqrt{\varepsilon}}$. *(ii)* In two dimensions, write $dn_{\nu_x\nu_y} = d\nu_x d\nu_y$ that in terms of $k$ becomes $dn_{k_xk_y} = \frac{L_xL_y}{\pi^2}dk_xdk_y$. Further, with the help of $k = \sqrt{2m\varepsilon}/\hbar$, you obtain $dn_\varepsilon = \frac{S}{2\pi}\sqrt{\frac{2m\varepsilon}{\hbar^2}}\left(\frac{2m}{\hbar^2}\right)^{1/2}\frac{d\varepsilon}{\sqrt{\varepsilon}} = \frac{S}{2\pi}\frac{2m}{\hbar^2}d\varepsilon$, and thus $\rho(\varepsilon) = \frac{S}{2\pi}\frac{2m}{\hbar^2}$, where $S = L_xL_y$ is the area of the rigid box. Note that in two dimensions the particle's density of states is a constant. The generalization for three dimensions is now straightforward and yields $\rho(\varepsilon) = \frac{V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}$, where $V = L_xL_yL_z$.

4. *(i)* Show that the variance of a uniform distribution of width $\Delta$ is $\Delta^2/12$. [*Hint:* Center the box at the origin.]

Solution: The uniform distribution is a box function centered at the origin of width $\Delta = 2a$, that is $f(x)\{^{(2a)^{-1},\ \text{if}\ |x|<a}_{0,\ \text{otherwise}}$. The variance is, $\text{Var}(x) = \frac{1}{2a}\int_{-a}^{+a}x^2dx = \frac{1}{2a}\,\frac{1}{3}x^3\Big|_{-a}^{+a} = \frac{1}{6a}2a^3 = \frac{\Delta^2}{12}$.