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Problems set # 7 **Physics 303** October 21, 2014

1. The distribution of particle speeds of a certain hypothetical gas is

$$N(v)dv = Ave^{-v/v_0}dv,$$

where A and v_0 are constants. *(i)* Determine A so that $f(v) \equiv N(v)/N$ is a true probability density function, i.e., $\int_0^x f(v)dv = 1$. *(ii)* Find $\bar{v}$ and $v_{\rm rms}$ in terms of v_0. *(iii)* Differentiate $f(v)$ with respect to v and set the result equal to zero to find the most probable speed v_m. *(iv* The standard deviation of the speeds from the mean is defined as

$$\sigma = \left[\overline{(v-\bar{v})^2}\right]^{1/2},$$

where the bar denotes the mean value. Show that

$$\sigma = \left[\overline{v^2} - (\bar{v})^2\right]^{1/2}$$

in general. What is σ for this problem?

Solution: *(i)* Write the normalization condition, $1 = \int_0^\infty \frac{N(v)}{N} dv = \int_0^\infty \frac{A}{N} v e^{-v/v_0} = \frac{A}{N} v_0^2$, to obtain $A = N/v_0$. *(ii)* $\bar{v} = \int_0^\infty dv\, v f(v) = \int dv\, v^2\, e^{-v/v_0}/v_0^2 = 2! v_0 = 2v_0$ and $v_{\rm rms} \equiv \bar{v^2} = \sqrt{\int_0^\infty dv v^2 f(v)} = \sqrt{\int_0^\infty dv v^3 e^{-v/v_0}/v_0^2} = \sqrt{3! v_0^2} = \sqrt{6} v_0$. *(iii)* $\left(\frac{df(v)}{dv}\right)_{v=v_m} \Rightarrow v_m = v_0$. *(iv)* Let X be a random variable with mean value μ, that is $E[X] = \mu$, where the operator E denotes the average or expected value of X. Then the standard deviation of X is given by $\sigma = \sqrt{E[(X-\mu)^2]} = \sqrt{E[X^2] + E[(-2\mu X)] + E[\mu^2]} = \sqrt{E[X^2] - 2\mu E[X] + \mu^2} = \sqrt{E[X^2] - 2\mu^2 + \mu^2} = \sqrt{E[X^2] - \mu^2} = \sqrt{E[X^2] - (E[X])^2}$ and so identifying $E[X]$ with $\bar{v}$, it follows that $\sigma = \sqrt{\overline{v^2} - (\bar{v})^2}$. For the particular case at hand, $\sigma = \sqrt{v_{\rm rms}^2 - \bar{v}^2} = \sqrt{2} v_0$.

2. At standard temperature and pressure the mean speed of hydrogen molecules is $1.70 \times 10^3\ {\rm m\,s^{-1}}$. What is the particle flux?

Solution: $\Phi = \frac{1}{4}\frac{N}{V}\bar{v} = \frac{1}{4}\bar{v}\frac{P}{4kT} = 1.14 \times 10^{28}\ {\rm m^{-2}\,s^{-1}}$.

3. A vessel is divided into two parts of equal volume by means of a plane partition, in the middle of which is a very small hole. Initially, both parts of the vessel contain ideal gas at a temperature of 300 K and low pressure P. The temperature of one-half of the vessel is then raised to 600 K while the temperature of the other half remains at 300 K. Determine the pressure difference in terms of P between the two parts of the vessel when steady conditions are achieved.

Solution At equilibrium, the net flux between the two parts is zero, *i.e.*, $\Phi_{1\to 2} = \Phi_{2\to 1}$, yielding $\Phi_1 = \frac{P_1}{\sqrt{2\pi m k T_1}} = \Phi_2 = \frac{P_2}{\sqrt{2\pi m k T_2}}$; therefore, $P_1/P_2 = \sqrt{T_1/T_2}$. Since the number of particles is conserved through the process, using the equation of state for ideal gas, $PV = NkT$, it follows that $PV = (N_1 + N_2)kT$. Equivalently, $P_1V/2 = N_1kT_1$ and $P_2V/2 = N_2kT_2$. Solving for P_1 and P_2 you straightforwardly obtain $P_1 = \frac{2\sqrt{2}P}{1+\sqrt{2}}$ and $P_2 = \frac{4P}{1+\sqrt{2}}$, which implies $\Delta P = P_2 - P_1 = \frac{4-2\sqrt{2}}{1+\sqrt{2}}P$.

4. Consider a gas of one kilomole of He atoms at $T = 300$ K and $P = 1$ atm. The mean energy is $\bar{\epsilon} = \frac{3}{2}kT \approx 6 \times 10^{-21}$ J. Estimate the number of atoms of this gas whose energy ϵ lies in an interval of width 10^{-22} J around this mean value. [*Hint:* use the Maxwell distribution. It is a bit simpler to write it in terms of ϵ for this problem.]

Solution: Substitute $\epsilon = mv^2/2$ into $N(v)dv = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} dv$, to obtain an expression for the number of atoms as a function of energy $N(\epsilon)d\epsilon = \frac{2}{\sqrt{\pi}} \frac{N}{(kT)^{3/2}} e^{-\epsilon/kT} \sqrt{\epsilon} d\epsilon$. Then $N(\bar{\epsilon})\Delta\epsilon = \frac{2}{\sqrt{\pi}} \frac{6.02\times10^{26}}{(1.381\times10^{-23}300)^{3/2}} e^{-6\times10^{-21}/(1.381\times10^{-23}300)} \sqrt{6\times10^{-21}} \times 10^{-22} \approx 4.6 \times 10^{24}$ atoms.

5. The local poison control center wants to know more about carbonmonoxide and how it spreads through a room. You are asked *(i)* to calculate the mean free path of a carbon monoxide molecule and *(ii)* to estimate the mean time between collisions. The molar mass of carbon monoxide is 28 g/mol. Assume that the CO molecule is traveling in air at 300 K and 1 atm, and that the diameter of both CO molecules and air molecules are 3.75×10^{-10} m.

Solution: *(i)* Define the mean free path l of the molecules as the distance they typically travel before colliding with another molecule. If a molecule is travelig with a high velocity relative to the velocities of an ensemble of identical particles with random locations you can single out this molecule and consider other molecules as non-moving, to conclude that the molecule under consideration will hit, on average, the first of other molecules that is within the cylinder of height l and cross section $\sigma \approx \pi d^2$, where d is the molecule diameter. The volume σl of this cylinder is exactly the volume per molecule $1/n$, yielding $l \approx \frac{1}{\sigma n}$. If, on the other hand, the velocities of the identical molecules are comparable, the following correction applies: $l \approx \frac{1}{\sqrt{2}\sigma n} = 6.53 \times 10^{-8}$ m. *(ii)* The average time between collisions is $\tau = v/l = 1.55 \times 10^{-10}$ s.

6. The escape speed at the surface of a planet of radius R is $v_e = \sqrt{2\tilde{g}R}$, where $\tilde{g}$ is the acceleration due to gravity at the surface of the planet. If the rms speed of a gas is greater than about 15% to 20% of the escape speed of a planet, virtually all the molecules of that gas will escape the atmosphere of the planet. *(i)* At what temperature is $v_{\rm rms}$ for O_2 equal to 15% of the escape speed of the Earth? *(ii)* At what temperature is $v_{\rm rms}$ for H_2 equal to 15% of the escape speed for Earth? *(iii)* Temperatures in the exosphere (upper atmosphere) reach 1000 K. How does this help account for the low abundance of hydrogen in Earth's atmosphere? *(iv)* Compute the temperatures for which the rms speeds of O_2 and H_2 are equal to 15% of the escape speed at the surface of the moon, where $\tilde{g}$ is about one-sixth of its value on Earth and the radius of the moon is $R = 1738$ km. How this account for the absence of an atmosphere on the moon? *(v)* The escape speed for gas molecules in the atmosphere of Jupiter is 60 km/s and the surface temperature of Jupiter is typically $-150°$. Calculate the rms speeds for H_2, O_2, and CO_2 at this temperature. Are H_2, O_2, and CO_2 likely to be found in the atmosphere of Jupiter? *(vi)* The escape speed for gas molecules in the atmosphere of Mars is 5.0 km/s and the surface temperature of Mars is typically

0°. Calculate the rms speeds for H_2, O_2, and CO_2 at this temperature. Are H_2, O_2, and CO_2 likely to be found in the atmosphere of Mars?

Solution: *(i)* The average energy of the molecules in a gas is directly proportional to the temperature of the gas $\bar{\epsilon} = \frac{1}{2} m\overline{v^2} = \frac{3}{2} kT$ and so the average speed of molecules in a gas as a function of temperature is $v_{\rm rms} = \sqrt{\overline{v^2}} = \sqrt{3kT/m}$. Now combine $v_{\rm es} = \sqrt{2gR_\oplus}$ with $v_{{\rm rms}_{\rm O_2}}$ to obtain $T = 0.15^2 2gR_\oplus M_{\rm O_2}/(3R) \sim 3600$ K. *(ii)* Duplicate the procedure of *(i)* to obtain $T \sim 225$ K. *(iii)* For $T = 1000$ K, the $v_{\rm rms} \gg v_{\rm es}$. *(iv)* $T_{\rm O_2} = 0.15^2 \frac{2}{6} gR_{moon} M_{\rm O_2}/3R \sim 164$ K and $T_{\rm H_2} = 0.15^2 \frac{2}{6} gR_{\rm Moon} M_{\rm H_2}/3R \sim 10$ K. The mean surface temperature of the moon is 107°C during the day and −153°C during the night, so all molecules of O_2 and H_2 escape. *(v)* $v_{{\rm rms}_{\rm H_2}} \sim 1.24$ km/s, $v_{{\rm rms}_{\rm O_2}} \sim 310$ km/s, $v_{{\rm rms}_{\rm CO_2}} \sim 264$ km/s, whereas $v_{20\%} \sim 12$ km/s. This implies that H_2, O_2 and CO_2 should be found in the atmosphere of Jupiter. *(vi)* $v_{{\rm rms}_{\rm H_2}} \sim 1.8$ km/s, $v_{{\rm rms}_{\rm O_2}} \sim 461$ km/s, $v_{{\rm rms}_{\rm CO_2}} \sim 393$ km/s, whereas $v_{20\%} \sim 1$ km/s. This implies that O_2 and CO_2 should be found in the atmosphere of Mars, but the molecules of H_2 escape.