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Problems set # 13 **Physics 303** December 9, 2014

1. Derive the density of states $\rho(\varepsilon)$ as a function of ε for a free electron gas in one-dimension. (Assume periodic boundary conditions or confine the linear chain to some length L). Then calculate the Fermi energy ε_F at zero temperature for an N electron system.

Solution: The energy particle is $\varepsilon = p^2/2m$. Therefore, $dp = \left(\frac{m}{2\varepsilon}\right)^{1/2} d\varepsilon$. Taking account of the two states spins, you get, $\rho(\varepsilon)d\varepsilon = \frac{2Ldp}{h} = \frac{L(2m)^{1/2}}{h\varepsilon^{1/2}}d\varepsilon$, or $\rho(\varepsilon) = \frac{L(2m/\varepsilon)^{1/2}}{h}$. At temperature $T = 0$ K, the electrons will occupy all the state whose energy is from 0 to the Fermi energy ε_F. Hence $N = \int_0^{\varepsilon_F} \rho(\varepsilon)d\varepsilon$, giving $\varepsilon_F = \frac{h^2}{2m}\left(\frac{N}{2L}\right)^2$.

2. Calculate the average energy per particle, ε, for a Fermi gas a $T = 0$, given that ε_F is the Fermi energy. Consider two cases separately, non-relativistic and relativistic.

Solution: For a non-relativistic particle, $p \ll mc$ (p is the momentum and m the mass), it follows that $\varepsilon = \frac{p^2}{2m}$. We have $\rho(\varepsilon) = \sqrt{\varepsilon} \cdot$ constant. Then $\overline{\varepsilon} = \frac{\int_0^{\varepsilon_F} \varepsilon \cdot \sqrt{\varepsilon} d\varepsilon}{\int_0^{\varepsilon_F} \sqrt{\varepsilon} d\varepsilon} = \frac{9}{5}\varepsilon_F$. For $p \gg mc$, we have $\varepsilon = pc$ and $\rho(\varepsilon) = \varepsilon^2 \cdot$ constant. Therefore, $\overline{\varepsilon} = \frac{\int_0^{\varepsilon_F} \epsilon \cdot \varepsilon^2 d\varepsilon}{\int_0^{\varepsilon_F} \varepsilon^2 d\varepsilon} = \frac{3}{4}\varepsilon_F$.

3. For a system of electrons, assumed non-interacting, show that the probability of finding an electron in a state with energy Δ above the chemical potential μ is the same as the probability of finding an electron absent from a state with energy Δ below μ at any given temperature T.

Solution: According to the Fermi distribution the probability for a level ε to be occupied is $f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$, so the probability for finding an electron at $\varepsilon = \mu + \Delta$ is $f(\mu + \Delta) = \frac{1}{e^{\beta\Delta}+1}$, and the probability for not finding electrons at $\varepsilon = \mu - \Delta$ is given by $1 - f(\mu - \Delta) = \frac{1}{e^{\beta\Delta}+1}$. The two probabilities have the same value as reqired.