Prof. Anchordoqui

Problems set # 10 **Physics 303** November 11, 2014

1. A model thermodynamic system in which the allowed non-degenerate states have energies $0, \varepsilon, 2\varepsilon, 3\varepsilon, \cdots$ consists of four particles with total energy $U = 6\varepsilon$. Identify the possible distribution of particles, evaluate $\Omega = \sum_k \omega_k$ and work out the average occupation numbers for the various energy levels. *(i)* When the particles are gaseous bosons; *(ii)* when the particles are gaseous fermions.

Solution: *(i)* For gaseous bosons,

$$\begin{aligned} U = 6\varepsilon \quad & = 0\varepsilon \times 3 + 1\varepsilon \times 0 + 2\varepsilon \times 0 + 3\varepsilon \times 0 + 4\varepsilon \times 0 + 5\varepsilon \times 0 + 6\varepsilon \times 1 \\ & = 0\varepsilon \times 2 + 1\varepsilon \times 1 + 2\varepsilon \times 0 + 3\varepsilon \times 0 + 4\varepsilon \times 0 + 5\varepsilon \times 1 + 6\varepsilon \times 0 \\ & = 0\varepsilon \times 2 + 1\varepsilon \times 0 + 2\varepsilon \times 1 + 3\varepsilon \times 0 + 4\varepsilon \times 1 + 5\varepsilon \times 0 + 6\varepsilon \times 0 \\ & = 0\varepsilon \times 1 + 1\varepsilon \times 2 + 2\varepsilon \times 0 + 3\varepsilon \times 0 + 4\varepsilon \times 1 + 5\varepsilon \times 0 + 6\varepsilon \times 0 \\ & = 0\varepsilon \times 2 + 1\varepsilon \times 0 + 2\varepsilon \times 0 + 3\varepsilon \times 2 + 4\varepsilon \times 0 + 5\varepsilon \times 0 + 6\varepsilon \times 0 \\ & = 0\varepsilon \times 1 + 1\varepsilon \times 1 + 2\varepsilon \times 1 + 3\varepsilon \times 1 + 4\varepsilon \times 0 + 5\varepsilon \times 0 + 6\varepsilon \times 0 \\ & = 0\varepsilon \times 0 + 1\varepsilon \times 3 + 2\varepsilon \times 0 + 3\varepsilon \times 1 + 4\varepsilon \times 0 + 5\varepsilon \times 0 + 6\varepsilon \times 0 \\ & = 0\varepsilon \times 1 + 1\varepsilon \times 0 + 2\varepsilon \times 3 + 3\varepsilon \times 0 + 4\varepsilon \times 0 + 5\varepsilon \times 0 + 6\varepsilon \times 0 \\ & = 0\varepsilon \times 0 + 1\varepsilon \times 2 + 2\varepsilon \times 2 + 3\varepsilon \times 0 + 4\varepsilon \times 0 + 5\varepsilon \times 0 + 6\varepsilon \times 0 \end{aligned} \tag{1}$$

$\omega_j = 1$; $\Omega = \sum \omega_j = 9$; $\overline{N}_i = \sum N_{ij} \frac{\omega_j}{\Omega}$; $\overline{N}_0 = \frac{4}{3}$, $\overline{N}_1 = 1$, $\overline{N}_2 = \frac{7}{9}$, $\overline{N}_3 = \frac{4}{9}$, $\overline{N}_4 = \frac{2}{9}$, $\overline{N}_5 = \frac{1}{9}$, $\overline{N}_6 = \frac{1}{9}$. *(ii)* For gaseous fermions, the occupation number cannot be larger then one for any state, $U = 6\varepsilon = 0\varepsilon \times 1 + 1\varepsilon \times 1 + 2\varepsilon \times 1 + 3\varepsilon \times 1 + 4\varepsilon \times 0 + 5\varepsilon \times 0 + 6\varepsilon \times 0$, $\Omega = 1$, $\overline{N}_- = \overline{N}_1 = \overline{N}_2 = \overline{N}_3 = 1$.

2. Consider a system of N distinguishable particles, at temperature T, with available energy levels ε_1 and $\varepsilon_2 > \varepsilon_1$ (only two available non-degenerate energy levels). *(i)* Determine the equilibrium values of the occupation numbers N_1 and N_2, and the energy U of the system, as a function of temperature. *(ii)* Show that the specific heat is given by

$$C_V = Nk \left(\frac{\Delta}{kT}\right)^2 \frac{e^{-\Delta/kT}}{(1 + e^{-\Delta/kT})^2},$$

where $\Delta = \varepsilon_2 - \varepsilon_1$. Examine the low temperature and high temperature behavior of C_V/Nk, and sketch it as a function of kT/Δ.

Solution: From $N_i = N g_i \frac{e^{\varepsilon_i/kT}}{Z}$ it follows that $N_1 = N \frac{e^{-\varepsilon_1/kT}}{e^{-\varepsilon_1/kT} + e^{-\varepsilon_2/kT}}$ and $N_2 = N \frac{e^{-\varepsilon_3/kT}}{e^{-\varepsilon_1/kT} + e^{-\varepsilon_2/kT}}$ $U = \sum \varepsilon_i = N \frac{\varepsilon_1 e^{-\varepsilon_1/kT} + \varepsilon_2 e^{-\varepsilon_2/kT}}{e^{-\varepsilon_1/kT} + e^{-\varepsilon_2/kT}} = N \frac{\varepsilon_1 e^{(\varepsilon_2 - \varepsilon_1)/kT} + \varepsilon_2}{e^{(\varepsilon_2 - \varepsilon_1)/kT} + 1} = N \frac{\varepsilon_1 + \varepsilon_2 e^{-(\varepsilon_2 - \varepsilon_1)/kT}}{1 + e^{-(\varepsilon_2 - \varepsilon_1)/kT}}$. See Fig. 1. *(ii)* $C_V = \left(\frac{\partial U}{\partial T}\right)_V = Nk \left(\frac{\Delta}{kT}\right)^2 \frac{e^{-\Delta/kT}}{(1 + e^{-\Delta/kT})^2}$.

3. Show that for a system of a large number N of bosons at very low temperatures (such that they are all in the non-degenerate lowest energy state $\varepsilon = 0$) the chemical potential varies with temperature according to

$$\lim_{T\to 0} \mu = -\frac{kT}{N}.$$

Solution: For $T \to 0$, almost all the particles are at the ground state for gaseous bosons, $N \approx N_0 = \frac{1}{e^{(0-\mu)/kT}-1} \approx \frac{1}{e^{-\mu/kT}-1}$. Then, $1 + \frac{1}{N} = e^{-\mu/kT}$ and so $\ln\left(1+\frac{1}{N}\right) = -\frac{\mu}{kT}$, yielding $\frac{1}{N} \approx -\frac{\mu}{kT}$, or equivalently $\mu = -\frac{kT}{N}$.

4. *(i)* For a system of localized distingushable oscillators, Boltzmann statistics applies. Show that the entropy S is given by

$$S = -k \sum_J N_j \ln\left(\frac{N_j}{N}\right).$$

(ii) Substitute the Boltmann distribution in the previous result to show that

$$S = \frac{U}{T} + Nk \ln Z.$$

(iii) Show that

$$S = Nk\left[\frac{\theta/T}{e^{\theta/T}-1} - \ln(1-e^{-\theta/T})\right],$$

where $\theta = \hbar\omega/k$. Examine the behavior of S as $T \to 0$ and $T \to \infty$. [*Hint:* For part *(i)*, use $\omega = \omega_{\rm MB}$ (keeping the $N!$ in the numerator) and use Stirlings approximation. Note that the result of part *(i)* can be written as $S = -Nk\sum_j f_j \ln f_j$, where $f_j = Nj/N$ is the probability of occupying level j.]

Solution: *(i)* For distinguishable particles it follows that $S = k\ln\left(\frac{N}{\prod N_i!}\right) \approx k[N\ln N - N - \sum_i(N_i \ln N_i - N_i)] = -k\sum N_i \ln\frac{N_i}{N}$. *(ii)* For Boltzmann distribution, $N_i = N\frac{1}{e^{(\varepsilon_i-\mu)/kT}}$; $S = k\sum N_i \ln\frac{N_i}{N} = k\sum_i\left[N_i \ln\frac{1}{e^{(\varepsilon_i-\mu)/kT}}\right] = k\sum\left[N_i\left(\frac{\varepsilon_i-\mu}{kT}\right)\right] = \frac{1}{T}\sum N_i\varepsilon_i - k\frac{\mu}{kT}\sum N_i = \frac{U}{T} - k(-\ln Z)N = \frac{U}{T} + Nk\ln Z$. *(iii)* $U = Nk\theta\left(\frac{1}{2} + \frac{1}{e^{\theta/T}-1}\right)$; $Z = \frac{e^{-\theta/2T}}{1-e^{-\theta/T}}$; $S = \frac{U}{T} + Nk\ln Z = \frac{Nk\theta}{T}\left(\frac{1}{2}+\frac{1}{e^{\theta/T}-1}\right) + Nk\ln\left(\frac{e^{-\theta/2T}}{1-e^{-\theta/T}}\right) = Nk\left[\frac{\theta/T}{e^{\theta/T}-1} - \ln(1-e^{-\theta/T})\right]$.

5. The partition function of an Einstein solid is

$$Z = \frac{e^{-\theta_{\rm E}/2T}}{1-e^{\theta_{\rm E}/T}},$$

where $\theta_{\rm E}$ is the Einstein temperature. Treat the crystaline lattice as an assembly of 3N distinguishable oscillators. *(i)* Calculate the Helmholtz function F. *(ii)* Calculate the entropy S. Show that the entropy approaches zero as the temperature goes to absolute zero. Show that at high temperatures, $S \approx 3Nk[1+\ln(T/\theta_{\rm E})]$. Sketch $S/3Nk$ as a function of $T/\theta_{\rm E}$.

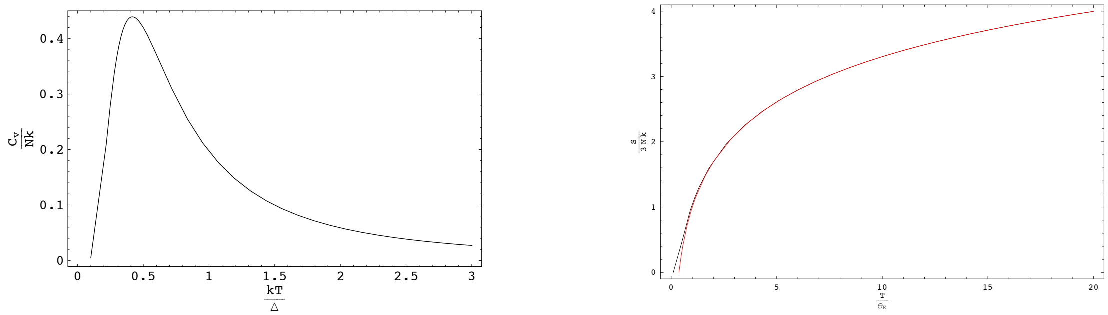

Figure 1: Problem 2 *(i)* (left) and problem 5 *(ii)* (right).

Solution: *(i)* $Z_{3N} = z^{3N}$; $F = -kT\ln Z_{3N} = -3NkT\ln z = -3NkT\left[-\frac{\theta_{\rm E}}{2T} - \ln(1-e^{-\theta_{\rm E}/T})\right]$. *(ii)* $S = -\left(\frac{\partial F}{\partial T}\right)_V = -3Nk\ln(1-e^{-\theta_{\rm E}/T}) + 3NkT\frac{\theta_{\rm E}}{T}\frac{e^{-\theta_{\rm E}/T}}{1-e^{-\theta_{\rm E}/T}}$, as the black line in Fig. 1. *(iii)* For $T \to 0$, $e^{-\theta_{\rm E}/T} \ll 1$, and so $S = -3Nk\ln(1-e^{-\theta_{\rm E}/T}) + 3NkT\frac{\theta_{\rm E}}{T^2}\frac{e^{-\theta_{\rm E}/T}}{1-e^{-\theta_{\rm E}/T}} \to -3Nk\ln 1 + 3Nk\frac{\theta_{\rm E}}{T}e^{-\theta_{\rm E}/T} \to 0$. For high temperature, $T \gg \theta_{\rm E}$ and $e^{-\theta_{\rm E}/T} \to 1 - \frac{\theta_{\rm E}}{T}$, yielding $S = -3Nk\ln(1-e^{-\theta_{\rm E}/T}) + 3NkT\frac{\theta_{\rm E}}{T^2}\frac{e^{-\theta_{\rm E}/T}}{1-e^{-\theta_{\rm E}/T}} \to -3Nk\ln(\theta_{\rm E}/T) + 3Nk$, as the red line in Fig. 1.