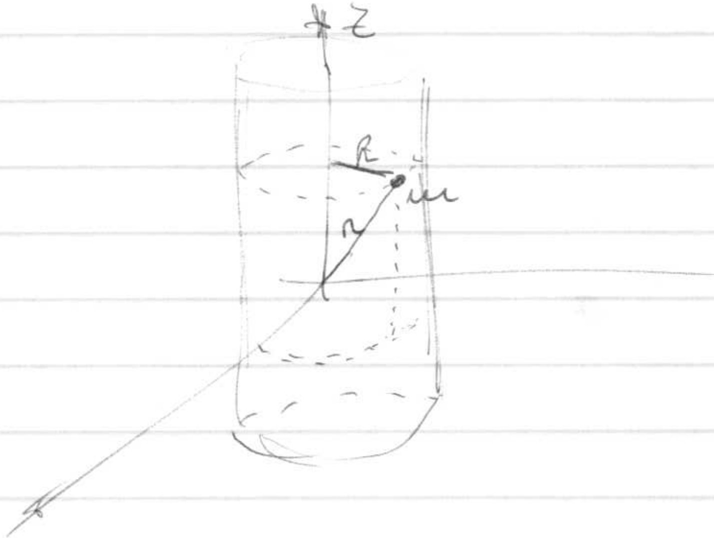
PROBLEMS # 14

1

(a)



$$U = \frac{1}{2} k r^2 = \frac{1}{2} k (x^2 + y^2 + z^2)$$

$$= \frac{1}{2} k (R^2 + z^2)$$

$$v^2 = \cancel{\dot{R}^2} + R^2 \dot{\theta}^2 + \dot{z}^2$$

but in this case R is a constant $\Rightarrow$

$$T = \frac{1}{2} m (R^2 \dot{\theta}^2 + \dot{z}^2)$$

we may now write the Lagrangian

as

$$L = T - U =$$

$$L = \frac{1}{2} m (R^2 \dot{\theta}^2 + \dot{z}^2) - \frac{1}{2} k (R^2 + z^2)$$

the generalized coordinates θ and z, and the generalized moments are

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = mR^2\dot\theta$$

$$p_z = \frac{\partial L}{\partial \dot z} = m\dot z$$

Because the system is conservative and because the equations of transformation between rectangular and cylindrical coordinates do not explicitly involve the time the Hamiltonian H is just the total energy expressed in terms of variables θ, p_θ, z, and p_z

$$H(z, p_\theta, p_z) = T + U$$

$$= \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}kz^2$$

where the constant $\frac{1}{2} k R^2$ has been suppressed. The equations of motion are therefore found from the canonical equations

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

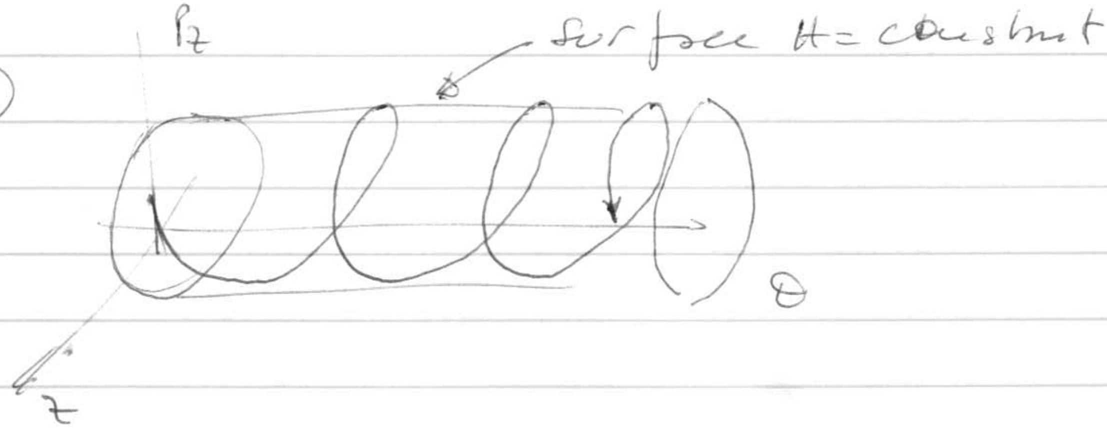
$p_\theta = mR^2\dot{\theta} =$ constant

$\ddot{z} + \omega_0^2 z = 0$

where $\omega_0^2 = k/m$

The motion in z is therefore simple harmonic

(b)



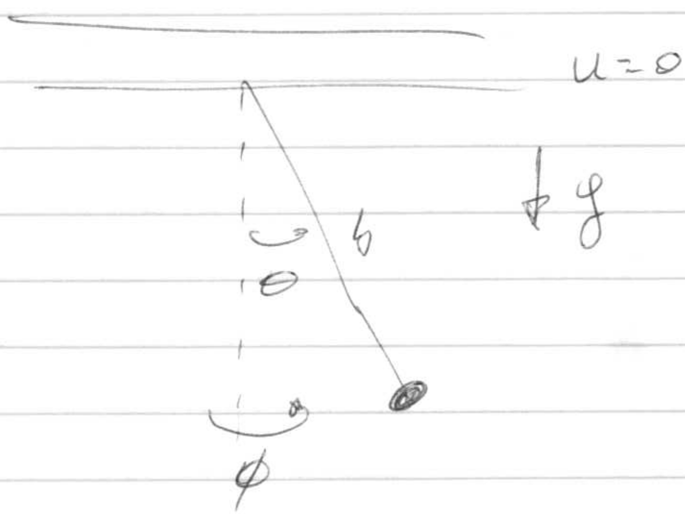
the particle has two degrees of freedom (θ, z), so the phase space for this example is actually four dimensional: θ, p_θ, z, p_z. But p_θ is a constant and therefore may be suppressed. In the z direction, the motion is simple harmonic, so the projection onto the $z-p_z$ plane of the phase space path for any total energy H is just an ellipse. Because $\dot{\theta}$ = constant, the phase path must represent the motion increasing uniformly with θ. Thus, the phase space on any surface H = constant is a uniform elliptical spiral.

(2) The generalized coordinates are θ and ϕ. The kinetic energy is

$$T = \frac{1}{2} m b^2 \dot{\theta}^2 + \frac{1}{2} m b^2 \sin^2\theta \, \dot{\phi}^2$$

The only force acting on the pendulum (other than at the point of support) is gravity, and we define the potential zero to be at the pendulum's point of attachment

$$U = -mgb\cos\theta$$



The generalized moments are then

$$P_\theta = \frac{\partial L}{\partial \dot\theta} = m b^2 \dot\theta \qquad (1)$$

$$P_\phi = \frac{\partial L}{\partial \dot\phi} = m b^2 \sin^2\theta \, \dot\phi \qquad (2)$$

We can solve eq (1) and (2) for $\dot\theta$ and $\dot\phi$ in terms of P_θ and P_ϕ.

$$H = T + U$$

$$= \frac{1}{2} m b^2 \frac{P_\theta^2}{(m b^2)^2} + \frac{1}{2} \frac{m b^2 \sin^2\theta \, P_\phi^2}{(m b^2 \sin^2\theta)^2} - m g b \cos\theta$$

$$= \frac{P_\theta^2}{2 m b^2} + \frac{P_\phi^2}{2 m b^2 \sin^2\theta} - m g b \cos\theta$$

The equations of motion are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mb^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mb^2 \sin^2\theta}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{mb^2 \sin^3\theta} - mgb \sin\theta$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

Because ϕ is cyclic, the momentum p_ϕ about the symmetry axis is constant.

(3) According to the equipartition theorem the average kinetic energy of each atom in the ideal gas is $\frac{3}{2} k T_1$, where k is the Boltzmann constant. The total average kinetic energy becomes

$$\langle T \rangle = \frac{3}{2} N k T_1$$

The right hand side of the virial theorem contains the forces F_α. For an ideal gas, no force of interaction occurs between atoms. The only force is represented by the force of constraint of the walls. The atoms bounce elastically of the walls, which are exerting a pressure on the atoms.

Because the pressure is force per unit area, we find the instantaneous differential force over a differential area to be

$$dF_\alpha = -\hat{n} P \, dA$$

where $\hat{n}$ is a unit vector normal to the surface dA and pointing outward. The right-hand side of the virial theorem becomes

$$-\frac{1}{2} \left\langle \sum_\alpha F_\alpha \cdot r_\alpha \right\rangle = \frac{P}{2} \int \hat{n} \cdot \hat{r} \, dA$$

we use the divergence theorem to relate the surface integral to a volume integral

$$\int \hat{n} \cdot \hat{r}\, dA = \int \bar{\nabla} \cdot \vec{r}\, dV = 3\int dV = 3V$$

The virial theorem result is

$$\frac{3}{2} NkT = \frac{3PV}{2}$$

$$NkT = PV$$

which is the ideal gas law

(4)

a) From the definition of a total derivative we can write

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left[\frac{\partial g}{\partial q_k}\frac{\partial q_k}{\partial t} + \frac{\partial g}{\partial p_k}\frac{\partial p_k}{\partial t} \right] \quad (1)$$

Using the canonical equations

$$\frac{\partial q_k}{\partial t} = \dot{q}_k = \frac{\partial H}{\partial p_k}$$

$$\frac{\partial p_k}{\partial t} = \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

we can write (1) as

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left[\frac{\partial g}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial g}{\partial p_k}\frac{\partial H}{\partial q_k} \right]$$

or $\frac{dg}{dt} = \frac{\partial g}{\partial t} + [g, H]$

b) $\dot{q}_j = \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p_j}$

According to the definition of

The Poisson brackets

$$[q_j, H] = \sum_k \left[\frac{\partial q_j}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial q_j}{\partial p_k}\frac{\partial H}{\partial q_k}\right] \quad (2)$$

but $\frac{\partial q_j}{\partial q_k} = \delta_{jk}$ and $\frac{\partial q_j}{\partial p_k} = 0$ for any j, k

then (2) can be expressed as

$$[q_j, H] = \frac{\partial H}{\partial p_j} = \dot{q}_j$$

In the same way, from the canonical equations

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$$[p_j, H] = \sum_k \left[\frac{\partial p_j}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial p_j}{\partial p_k}\frac{\partial H}{\partial q_k}\right]$$

but

$$\frac{\partial p_j}{\partial p_k} = \delta_{jk} \quad \text{and} \quad \frac{\partial p_j}{\partial q_k} = 0 \quad \text{for any } j,k$$

$$\Rightarrow \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} = [p_j, H]$$

c) $$[p_k, p_j] = \sum_\ell \left[\frac{\partial p_k}{\partial q_\ell} \frac{\partial p_j}{\partial p_\ell} - \frac{\partial p_k}{\partial p_\ell} \frac{\partial p_j}{\partial q_\ell} \right] \quad (3)$$

Since $\frac{\partial p_k}{\partial q_\ell} = 0$ for any k, ℓ

the right hand side of (3) vanishes

and $[p_k, p_j] = 0$

In the same way

$$[q_k, q_j] = \sum_\ell \left[\frac{\partial q_k}{\partial q_\ell} \frac{\partial q_j}{\partial p_\ell} - \frac{\partial q_k}{\partial p_\ell} \frac{\partial q_j}{\partial q_\ell} \right]$$

since $\frac{\partial q_j}{\partial p_l} = 0$ for any j, l

the r.h.s. vanishes and

$$[q_k, q_j] = 0$$

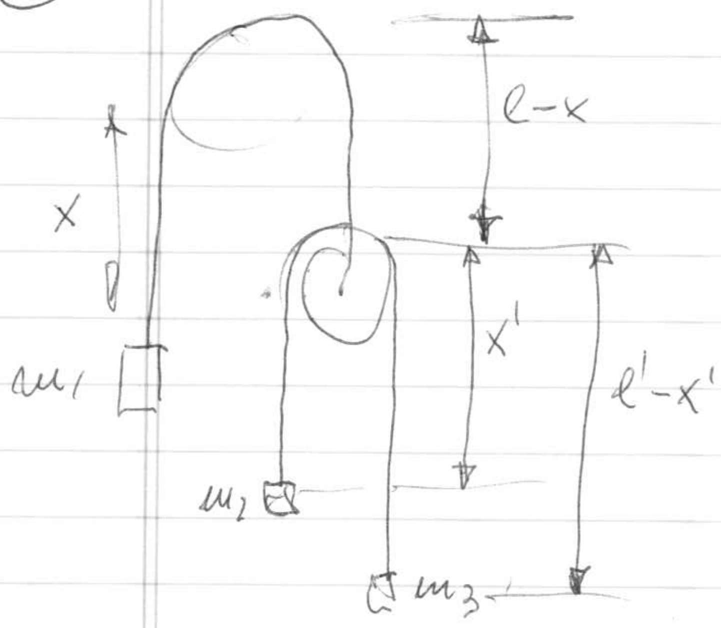
(d) $[q_k, p_j] = \sum_l \left[\frac{\partial q_k}{\partial q_l}\frac{\partial p_j}{\partial p_l} - \frac{\partial q_k}{\partial p_l}\frac{\partial p_j}{\partial q_l}\right]$

$$= \sum_l \delta_{kl}\,\delta_{jl}$$

so $[q_k, p_j] = \delta_{kj}$

(e) Let $f(p_k, q_k)$ be a quantity that does not depend explicitly on time. If $f(p_k, q_k)$ commutes with H, i.e., if $[f, H] = 0$ then according to the result in (a) $\frac{df}{dt} = 0$ and f is a constant of motion.

(5)



Neglect the masses of the pulleys

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left(\dot{x}' - \dot{x}\right)^2 + \frac{1}{2} m_3 \left(-\dot{x} - \dot{x}'\right)^2$$

$$U = -m_1 g x - m_2 g \left(\ell - x + x'\right) - m_3 g \left(\ell - x + \ell' - x'\right)$$

$$L = \frac{1}{2}\left(m_1 + m_2 + m_3\right)\dot{x}^2 + \frac{1}{2}\left(m_2 + m_3\right)\dot{x}'^2 + \dot{x}\dot{x}'\left(m_3 - m_2\right) + g\left(m_1 - m_2 - m_3\right)x + g\left(m_2 - m_3\right)x'$$

$+$ Const.

We redefine the zero in U such that the constant in $L = 0$

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2 + m_3)\,\dot{x} + (m_3 - m_2)\,\dot{x}' \qquad (1)$$

$$p_{x'} = \frac{\partial L}{\partial \dot{x}'} = (m_3 - m_2)\,\dot{x} + (m_2 + m_3)\,\dot{x}' \qquad (2)$$

Solving (1) and (2) for p_x and $p_{x'}$ gives

$$\dot{x} = D^{-1}\left[(m_2 + m_3)\,p_x + (m_2 - m_3)\,p'_x\right]$$

$$\dot{x}' = D^{-1}\left[(m_2 - m_3)\,p_x + (m_1 + m_2 + m_3)\,p_{x'}\right]$$

~~where $D =$ [illegible] $D = m_1 m_3 + m_1 m_2$~~

where $D = m_1 m_3 + m_1 m_2 + 4 m_2 m_3$

$$H = T + U$$

$$= \frac{1}{2}(m_1 + m_2 + m_3)\,\dot{x}^2 + \frac{1}{2}(m_2 + m_3)\,\dot{x}'$$

$$+ (m_3 - m_2)\,\dot{x}\,\dot{x} - g(m_1 - m_2 - m_3)\,x$$

$$- g(m_2 - m_3)\,x'$$

Substituting for $\dot{x}$ and $\dot{x}'$ and simplifying gives.

$$H = \frac{1}{2}(m_2+m_3)D^{-1}P_x^2 + \frac{1}{2}(m_1+m_2+m_3)$$

$$D^{-1}P_{x'}^2 + (m_2-m_3)D^{-1}P_xP_{x'} -$$

$$g(m_1-m_2+m_3)x - g(m_2-m_3)x'$$

The equation of motion are

$$\dot{x} = \frac{\partial H}{\partial P_x} = (m_2+m_3)D^{-1}P_x + (m_2-m_3)D^{-1}P_{x'}$$

$$\dot{x}' = \frac{\partial H}{\partial P_{x'}} = (m_2-m_3)D^{-1}P_{x'} + (m_1+m_2+m_3)D^{-1}P_{x'}$$

$$\dot{P}_x = -\frac{\partial H}{\partial x} = g(m_1-m_2-m_3)$$

$$\dot{P}_{x'} = -\frac{\partial H}{\partial x'} = g(m_2-m_3)$$

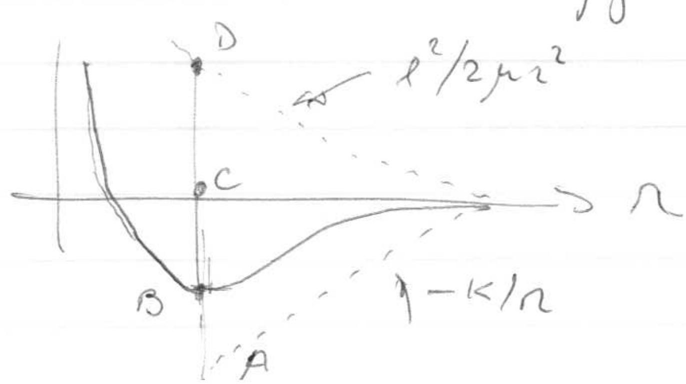
(e) When $k \to k/2$, the potential energy will decrease to half its former value, but the kinetic energy will remain the same. Since the original orbit is circular, the instantaneous values of T and U are equal to the average values, $\langle T \rangle$, and $\langle U \rangle$. For a $1/r^2$ force, the virial theorem states

$$\langle T \rangle = -\frac{1}{2} \langle U \rangle$$

Hence

$$E = T + U = -\frac{1}{2} U + U = \frac{1}{2} U$$

Now consider the energy diagram



where

$\overline{CB} = E$ original total energy

$\overline{CA} = U$ original potential energy

$\overline{CD} = V_c$ original centrifugal energy

The point B is obtained from $\overline{CB} = \overline{CA} - \overline{CD}$

According to the virial theorem

$E = \frac{1}{2} U$ or $\overline{CB} = 1/2\, \overline{CA}$. Therefore

$$\overline{CD} = \overline{CB} = \overline{BA}$$

Hence if U suddenly is halved, the total energy is raised from B by an amount equal to $(1/2)\overline{CA}$ or by $\overline{CB}$. Thus the total energy point is raised from B to C, i.e., $E_{final} = 0$ and the orbit is parabolic

(7)

The initial volume of phase space available to the beam is

$$V_0 = \pi R_0^2 \pi P_0^2$$

After focusing the volume in phase space is

$$V_1 = \pi R_1^2 \pi P_1$$

where now P_1 is the resulting radius of the distribution of transverse momentum components of the beam with a circular cross section of radius R_1.
From Liouville theorem the phase space accessible to the ensemble is invariant $\Rightarrow$

$$V_0 = \pi R_0^2 \pi P_0^2 = V_1 = \pi R_1^2 \pi P_1^2$$

$$\Rightarrow P_1 = \frac{R_0 P_0}{R_1}$$

If $R_1 < R_0$, then $P_1 > P_0$, which means that the resulting spread in momentum distribution has increased.

This result means that when the beam is better focussed, the transverse momentum components are increased and there is a subsequent divergence of the beam past the point of focus.

(8)

We use z_i and p_j as our generalized coordinates, the subscript ~~ith~~ corresponding to the ith particle. For a uniform field in the z direction the trajectories $z = z(t)$ and moments $p = p(t)$ are given by

$$z_i = z_0 + v_{i0} t - \frac{1}{2} g t^2$$

$$p_i = p_{0i} - m g t$$

where z_{i0}, p_{i0}, and $v_{i0} = p_{i0}/m$ are the initial displacement, momentum, and velocity of the ith particle

$$z_1 = z_0 + \frac{p_0 t}{m} - \frac{1}{2} g t^2$$

$$p_1 = p_0 - m g t$$

$$z_2 = z_0 + \Delta z_0 + \frac{p_0 t}{m} - \frac{1}{2} g t^2$$

$$p_2 = p_0 - m g t$$

$$z_3 = z_0 + \frac{(p_0 + \Delta p_0)t}{m} - \frac{1}{2} g t^2$$

$$p_3 = p_0 + \Delta p_0 - m g t$$

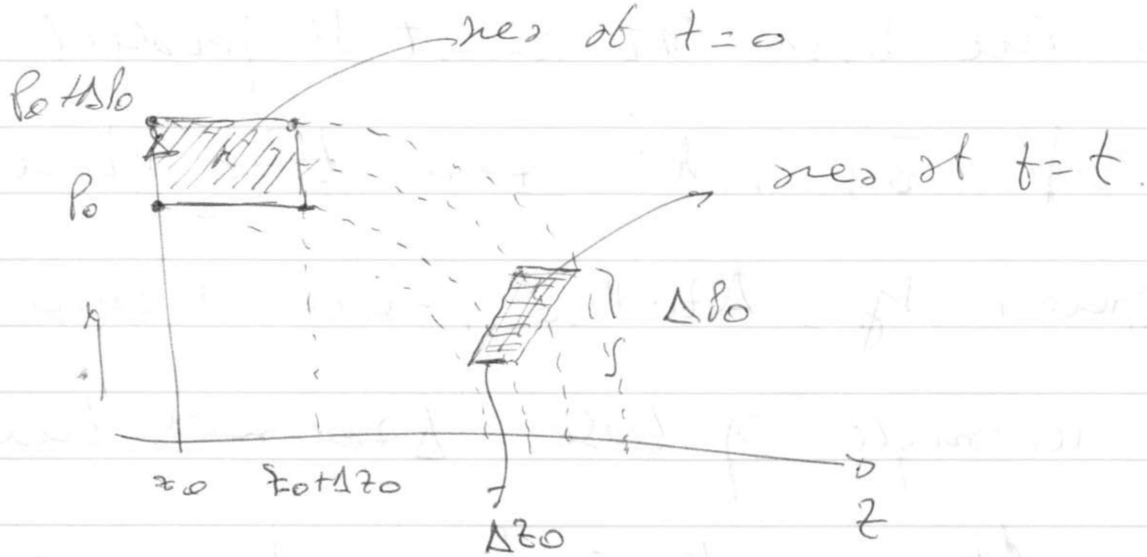
$$z_4 = z_0 + \Delta z_0 + \frac{(p_0 + \Delta p_0)t}{m} - \frac{1}{2} g t^2$$

$$p_4 = p_0 + \Delta p_0 - m g t$$

The Hamiltonian particle corresponding to the ith particle is

(3) $H_i = T_i + V_i = \frac{m \dot{z}_i^2}{2} + m g z_i = \frac{p_i^2}{2m} + m g z_i = ct.$

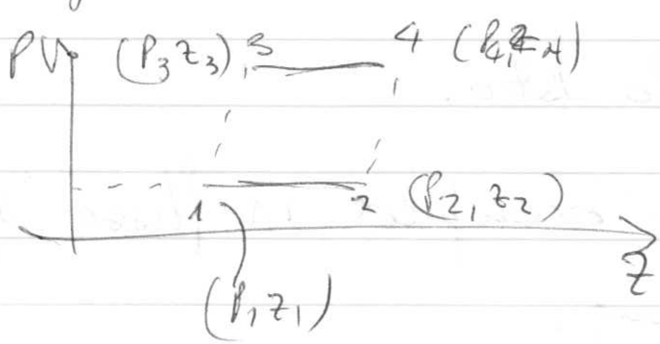
From (3) the phase space particle is a parabola as shown in the figure



From this diagram (as well as from the eq. of motion) it can be seen that for any time t

$$p_1 = p_2 \quad \text{and} \quad p_3 = p_4$$

Then for a certain time t the area described by the representative points will be of general form



where the base $\overline{12}$ must be parallel to the top $\overline{34}$. At time $t=0$ the area is given by $\Delta z_0 \Delta p_0$, since it corresponds to a rectangle of base Δz_0 and height Δp_0. At any other time the area will be

$$A = \{\text{base of parallelogram}\big|_{t=t_1} \times \text{height of parallelogram}\big|_{t=t_1}\}$$

$$\text{base of parallelogram}\big|_{t=t_1} = (z_2 - z_1)\big|_{t=t_1} = (z_4 - z_3)\big|_{t=t_1} = \Delta z_0$$

$$\text{height of parallelogram}\big|_{t=t_1} = (p_3 - p_1)\big|_{t=t_1} = (p_4 - p_2)\big|_{t=t_1} = \Delta p_0$$

$$\Rightarrow A = \Delta p_0 \Delta z_0.$$

Thus the area occupied in phase space is a constant in time.

(9) By the virial theorem $T = -U/2$ for a circular orbit. The firing of the rocket doesn't change U, so

$U_f = U_i$

But

$$T_f = \frac{1}{2} m \left(v^2 + v^2\right) = 2T_i$$

So $E_f = 2T_i + U_i = -U_i + U_i = 0$

$$\frac{E_f}{E_i} = 0$$

The firing of the rocket doesn't change the angular momentum since it fires in the radial direction $\frac{l_f}{l_i} = 1$

(5) $E=0$ means the orbit is parabolic. The satellite will be lost

$$E(r)=0 \qquad U(r) = -\frac{GM_\oplus m_s}{r}$$

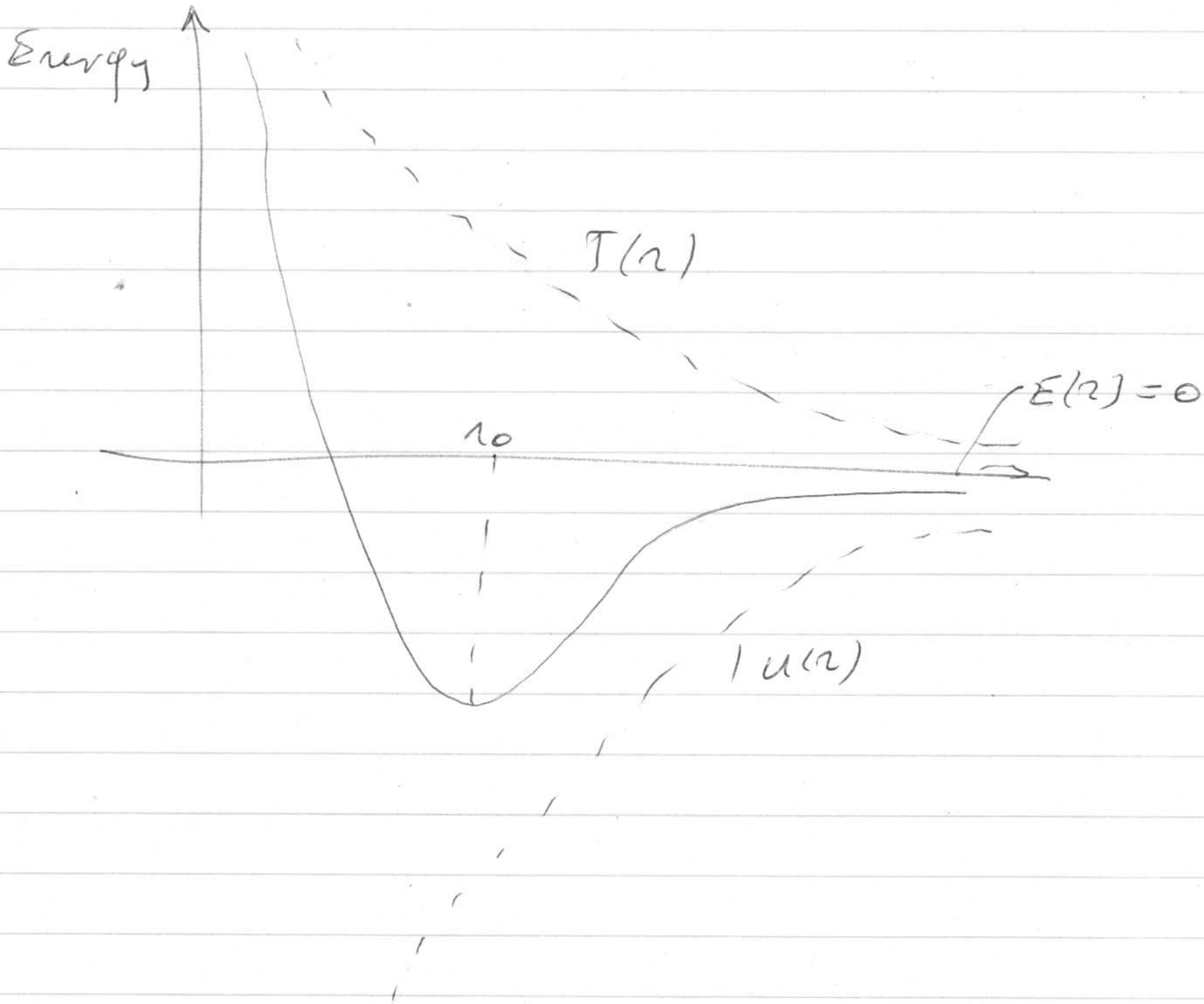
$$T(r) = E-U = \frac{GM_\oplus m_s}{r}$$

$$V(r) = U(r) + \frac{\ell^2}{2\mu r^2} = -\frac{GM_\oplus m_s}{r} + \frac{\ell^2}{2\mu r^2}$$

Behavior of $V(r)$ is determined by

$\ell^2/2\mu r^2$ for small r

$-\frac{GM_\oplus m_s}{r}$ for large r



Minimum in V(r) is found by setting

$$\frac{dV}{dr} = 0 \text{ at } r = r_0$$

$$0 = \frac{G M_\oplus m_S}{r_0^2} - \frac{\ell^2}{\mu r_0^3}$$

$$r_0 = \frac{\ell^2}{\mu G M_\oplus m_S}$$