PROBLEMS # 12

(1) For an elastic collision between two objects of equal mass, with the target object initially stationary, the angle between the final velocity of the objects is 90°. Here is a proof of that

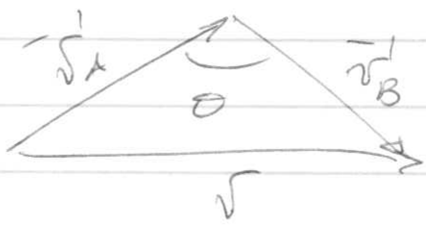
$$m\vec{v} = m\vec{v}_A' + m\vec{v}_B' \Rightarrow \vec{v} = \vec{v}_A' + \vec{v}_B'$$

Kinetic energy conservation says

$$\frac{1}{2} m v^2 = \frac{1}{2} m v_A'^2 + \frac{1}{2} m v_B'^2$$

$\Rightarrow$

$$v^2 = v_A'^2 + v_B'^2$$



Applying law of cosines

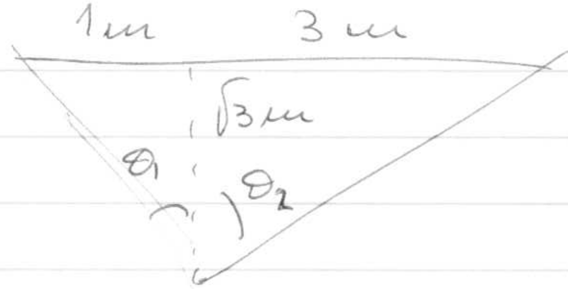
$$v^2 = v_A'^2 + v_B'^2 - 2 v_A' v_B' \cos\theta$$

Equating the two expressions for v^2 gives

$$v_A'^2 + v_B'^2 - 2 v_A' v_B' \cos\theta = v_A'^2 + v_B'^2$$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = 90^\circ$$

Let's assume that the target ball is hit "correctly" so that it goes in the pocket



We find θ_1 from the geometry of the left triangle

$$\theta_1 = \text{ARCTAN } \frac{1}{\sqrt{3}} = 30^\circ$$

We can find θ_2, the required angle to go in the pocket from the geometry of the right triangle

$$\theta_2 = \text{Actan } \frac{3}{\sqrt{3}} = 60^\circ$$

Since the balls separate at an angle of 90° (in an elastic collision), if the target ball goes in the pocket, this does appear to be a good possibility of a scratch shot

(2)

$$u(r) = 0 \qquad \text{for } r > a$$

$$u(r) = \infty \qquad \text{for } r \leq a$$

This is the interaction of impenetrables spheres which only exert a force on one another when they are in physical contact (eg billiard balls). If the particles of the first species have radius R_1, and the particles of the second species have radius R_2 then $a = R_1 + R_2$. In other words the centers of the two particles, one from either species, can never be less than distance a, where a is the sum of their radii

$$\Theta = \int_0^{u_{max}} \frac{b\,du}{\sqrt{1 - b^2u^2 - U(u)/E}}$$

$u_{max} = 1/r_{min}$ $\qquad E = T_0^*$

$$\Theta = \pi - 2\int_0^{1/a} \frac{b\,du}{\sqrt{1-b^2u^2}} = \pi - 2\arcsin(b/a)$$

The above formula can be re-arranged to give

$$b(\theta) = a\cos(\theta/2)$$

$$\sigma(\theta) = \frac{b}{\sin\theta}\left(\frac{db}{d\theta}\right)$$

$$\frac{d\sigma(\theta)}{d\Omega} = \frac{1}{\sin\theta} \, a \cos\frac{\theta}{2} \, \frac{a}{2} \sin\frac{\theta}{2} = \frac{a^2}{4}$$

$$\sigma = \int \frac{d\sigma}{d\Omega} \, d\Omega = \pi a^2$$

obviously this result makes a lot of sense the total scattering cross section of impenetrable spheres is simply the area of a circle whose radius is the sum of the radii of the two types of particles on the two beams.

(3) The total number of target nucleons present in $1 \, Km^3$ of ice is

$$N_T = \rho_{ice} \, N_A \, V \sim 6 \times 10^{38}$$

where $\rho_{ice} = 1\ g/cm^3$ is the density of ice, $N_A = 6.022 \times 10^{23}\ mol^{-1}$ is Avogadro's number and $V = 10^{15}\ cm^3$ is the total volume.

Assuming the Earth is completely opaque to the propagation of neutrinos (i.e. an effective aperture for detection at IceCube of 2π sr) the total number of events expected in $T = 15$ yr is

$$N = 2\pi\, N_T\, T \int_{10^8\,\mathrm{GeV}}^{10^{11}\,\mathrm{GeV}} \Phi(E_\nu)\, \sigma(E_\nu)\, dE_\nu \simeq 7$$

(4)

The number of recoiled particles scattered into unit solid angle in each of the two systems, lab & CM are the same

$$\sigma(\phi) \sin\phi \, d\phi = \sigma(\xi) \sin\xi \, d\xi \qquad (1)$$

where ϕ and ξ are the CM and lab angles of the recoil particle. From (1) we can write

$$\frac{\sigma(\phi)}{\sigma(\xi)} = \frac{\sin\xi \, d\xi}{\sin\phi \, d\phi} \qquad (2)$$

It is easily seen that for elastic collisions $2\xi = \phi$ [See eq (A20) in the appendix]

Hence

$$\frac{\sin\xi}{\sin\phi} = \frac{\sin\xi}{\sin 2\xi} = \frac{1}{2\cos\xi} \qquad (3)$$

and $$\frac{d\xi}{d\phi} = \frac{1}{2} \qquad (4)$$

Using (3) and (4) in (2), we have

$$\frac{\sigma(\phi)}{\sigma(\xi)} = \frac{1}{4\cos\xi} \qquad (5)$$

For $m_1 = m_2$ from eq (A24) we have $T_0^* = \frac{1}{2} T_0 \Rightarrow$ the Rutherford scattering formula

$$\sigma(\theta) = \frac{k^2}{(4T_0^*)^2} \frac{1}{\sin^4(\theta/2)} \qquad (6)$$

becomes

$$\sigma(\theta) = \frac{k^2}{4T_0^2} \frac{1}{\sin^4(\theta/2)} \qquad (7)$$

For this case we also have $\psi = \theta/2$ and $\psi = \pi/2 - \xi$ [See eqs (A15) and (A21)]

hence

$$\sin\frac{\theta}{2} = \sin\psi = \sin(\pi/2 - \xi) = \cos\xi$$

and since the CM recoil cross section

$\sigma(\phi)$ is the same as the CM

scattering cross section $\sigma(\theta)$, (7)

becomes

$$\sigma(\phi) = \frac{k^2}{4T_0^2} \frac{1}{\cos^4 \xi}$$

Using (5) to express $\sigma(\xi)$, we obtain

$$\sigma(\xi) = \sigma(\phi) \quad 4\cos\xi$$

$$\sigma(\xi) = \frac{K^2}{T_0^2} \frac{1}{\cos^3 \xi}$$

(5) In the case $m_1 >> m_2$ the scattering angle ψ for the incident particle measured in the LAB system is very small for all energies.

We can then anticipate that $\sigma(\psi)$ will rapidly approach zero as ψ increases.

The Rutherford cross section in terms of the scattering angle in the CM reads

$$\sigma_{CM}(\theta) = \frac{K^2}{(4T_0^*)^2} \frac{1}{\sin^4(\theta/2)} \qquad (1)$$

For $m_1 >> m_2$, from (A23 $\frac{1}{2}$)

$$T_0^* = \frac{m_2}{m_1+m_2} T_0 \simeq \frac{m_2}{m_1} T_0 \qquad (2)$$

Using (A12)

$$\tan\psi = \frac{\sin\theta}{\frac{m_1}{m_2} + \cos\theta} \simeq \frac{m_2}{m_1} \sin\theta \quad (3)$$

and thus, since ψ is expected to be small for all the cases of interest

$$\sin\theta \simeq \frac{m_1}{m_2} \tan\psi \simeq \frac{m_1}{m_2} \psi \quad (4)$$

Then

$$\cos\theta = \sqrt{1 - \left[\frac{m_1}{m_2}\psi\right]^2} \quad (5)$$

and

$$\sin^2(\theta/2) = \frac{1}{2}\left[1 - \sqrt{1 - \left[\frac{m_1}{m_2}\psi\right]^2}\right] \quad (6)$$

Note that $\psi \ll 1$, but since $m_1 \gg m_2$, the quantity $m_1 \psi / m_2$ is not necessarily small compared to unity. Now with the help of (2) and (6), we can re-write the CM cross section in terms of ψ as

$$\sigma_{CM}(\psi) = \left[\frac{m_1 K}{2 m_2 T_0}\right]^2 \frac{1}{\left[1-\sqrt{1-\left[\frac{m_1 \sin\psi}{m_2}\right]^2}\,\right]^2} \qquad (7)$$

Because the total number of particles scattered into a unit solid angle must be the same in the Lab system as in the CM system

$$\sigma(\theta)\, d\Omega^* = \sigma(\psi)\, d\Omega \qquad (8)$$

$$\sigma(\theta) \cdot 2\pi \sin\theta\, d\theta = \sigma(\psi)\, 2\pi \sin\psi\, d\psi \qquad (9)$$

$$\therefore \sigma(\psi) = \sigma(\theta)\, \frac{\sin\theta\, d\theta}{\sin\psi\, d\psi} \qquad (10)$$

Using the sin law

$$\frac{\sin(\theta-\psi)}{\sin\psi} = \frac{m_1}{m_2} \equiv x \qquad (11)$$

Differentiating this equation

$$\frac{d\theta}{d\psi} = \frac{\sin(\theta-\psi)\cos\psi}{\cos(\theta-\psi)\sin\psi} + 1 \qquad (12)$$

Expanding $\sin(\theta-\psi)$ and simplifying we have

$$\frac{d\theta}{d\psi} = \frac{\sin\theta}{\cos(\theta-\psi)\sin\psi} \qquad (13)$$

and so

$$\sigma(\psi) = \sigma(\theta)\, \frac{\sin^2\theta}{\cos(\theta-\psi)\sin^2\psi} \qquad (14)$$

Multiplying both sides of Eq (11) by $\cos\psi$ and then adding $\cos(\theta-\psi)$ to both sides we have

$$\frac{\sin(\theta-\psi)\cos\psi}{\sin\psi} + \cos(\theta-\psi) = x\cos\psi + \cos(\theta-\psi)$$

Expanding $\sin(\theta-\psi)$ and $\cos(\theta-\psi)$ on the left-hand side we obtain

$$\frac{\sin\theta}{\sin\psi} = x\cos\psi + \cos(\theta-\psi)$$

Substituting this result into (14)

$$\sigma(\psi) = \sigma(\theta)\,\frac{\left[x\cos\psi + \cos(\theta-\psi)\right]^2}{\cos(\theta-\psi)}$$

and from eq (11) we have

$$\cos(\theta-\psi) = \sqrt{1 - x^2\sin^2\psi}$$

Hence

$$\sigma(\psi) = \sigma(\theta)\,\frac{\left[x\cos\psi + \sqrt{1-x^2\sin^2\psi}\right]^2}{\sqrt{1-x^2\sin^2\psi}}$$

Then using eq (7)

$$\sigma_{LAB}(\psi) = \frac{\left[\dfrac{m_1 K}{2 m_2 T_0}\right]^2 \left[\dfrac{m_1}{m_2} + \sqrt{1 - \left[\dfrac{m_1 \psi}{m_2}\right]^2}\right]^2}{\sqrt{1 - \left(\dfrac{m_1}{m_2}\psi\right)^2}\left[1 - \sqrt{1 - \left[\dfrac{m_1}{m_2}\psi\right]^2}\right]^2} \qquad (8)$$

and so

$$\sigma_{LAB}(\psi) = \frac{\left[m_1^2 K / (2 m_2^2 T_0)\right]^2}{\left[1 - \sqrt{1 - \left[\dfrac{m_1 \psi}{m_2}\right]^2}\right]^2 \sqrt{1 - \left[\dfrac{m_1 \psi}{m_2}\right]^2}} \qquad (9)$$

This expression shows that the cross section has a second order divergence at $\psi = 0$. For values of $\psi > m_2/m_1$, Eq (8) gives complex values for σ_{LAB}. This result is due to the approximations involved in ~~the~~ its derivation, making this result invalid for angles larger than m_2/m_1.

(6) The differential cross section for Rutherford scattering in the CM is

$$\sigma(\theta) = \frac{K^2}{16 T_0^{*2}} \frac{1}{\sin^4\theta/2}$$

where $T_0^* = \frac{m_2}{m_1+m_2} T_0$ (see A 23½)

Thus, $\sigma(\theta) = \frac{K^2}{16 T_0^2} \frac{1}{\sin^4\theta/2} \left[\frac{m_1+m_2}{m_2}\right]^2$

$$= \frac{K^2}{16 T_0^2} \frac{1}{\sin^4\theta/2} \left[1 + \frac{m_1}{m_2}\right]^2$$

Since $m_1/m_2 \ll 1$, we expand

$$\left[1 + m_1/m_2\right]^2 \simeq 1 + 2\, m_1/m_2 + \dots$$

Thus, to 1st order on m_1/m_2, we have

$$\sigma(\theta) = \frac{K^2}{16 T_0^2} \frac{1}{\sin^4\theta/2} \left[1 + 2\frac{m_1}{m_2}\right]$$

(7) The potential for the given force law is $U(r) = \frac{k}{2r^2}$

First we make a change of variable $z = 1/r$. Then

$$\Theta = \int_0^{z_{max}} \frac{b\,dz}{\sqrt{1 - \left[b^2 + \frac{k}{mu_0^2}\right] z^2}}$$

$$= \frac{b}{\sqrt{b^2 + \frac{k}{mu_0^2}}} \arcsin \frac{z}{z_{max}} \Bigg|_0^{z_{max}}$$

$$z_{max} = \left[b^2 + \frac{k}{m u_0^2} \right]^{-1/2}$$

$$\Theta = \frac{\pi b}{2\sqrt{b^2 + k/m u_0^2}}$$

solving (2) for $b = b(\Theta)$

$$b(\Theta) = \sqrt{\frac{k}{m u_0^2}} \frac{2\Theta}{\sqrt{\pi^2 - 4\Theta^2}}$$

Now, since $\Theta = \frac{1}{2}(\pi - \theta)$

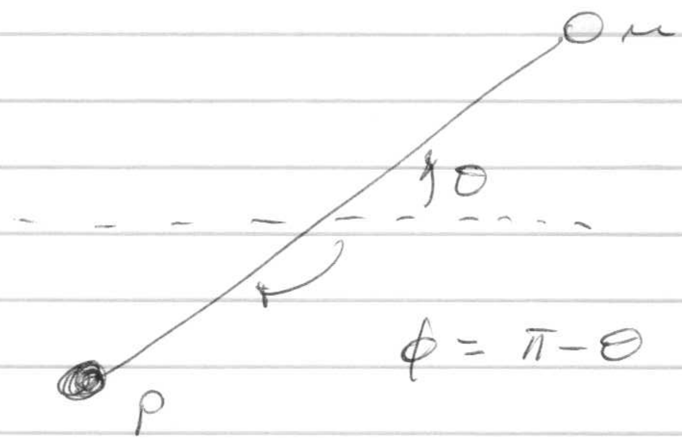
$$b(\theta) = \sqrt{\frac{k}{m u_0^2}} \frac{\pi - \theta}{\sqrt{\theta(2\pi - \theta)}}$$

The differential cross section can now be computed

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\sigma(\theta) = \frac{K\pi^2(\pi-\theta)}{m u_0^2 \theta^2 (2\pi-\theta)^2 \sin\theta}$$

(8)



In the CM system, whenever the neutron is scattered through the angle θ, the proton recoils at the angle $\phi = \pi - \theta$. Thus, the neutron scattering cross section is equal to the recoil cross section of the corresponding angles

$$\frac{dN_\mu}{d\Omega(\theta)} = \frac{dN_p}{d\Omega(\phi)}$$

Thus $$\frac{dN_\mu}{d\Omega(\theta)} = \frac{dN_p}{dT_p}\left|\frac{dT_p}{d\Omega(\phi)}\right|$$

where $\frac{dN_p}{dT_p}$ is the energy distribution

of the recoil protons. According

to the experiment $\frac{dN_p}{dT_p}$ = constant

Since $m_p \simeq m_\mu$, T_p is express in terms

of the angle ψ as [see eq. (A38) for details]

$$T_p = T_0 \sin^2\psi$$

We also have $\psi = \frac{\theta}{2}$ for the case

$m_p \simeq m_\mu$ [see eq (A15)]

Thus

$$\frac{dT_p}{d\Omega(\phi)} = \frac{1}{2\pi \sin\phi} \frac{d}{d\phi}\left(T_0 \sin^2\psi\right)$$

or
$$\left|\frac{dT_p}{d\Omega(\phi)}\right| = \left|\frac{T_0}{2\pi \sin\phi} \frac{d}{d\phi} \sin^2\left[\frac{\pi-\phi}{2}\right]\right|$$

$$= \frac{T_0}{2\pi \sin\phi} \sin\frac{\phi}{2} \cos\frac{\phi}{2} = \frac{T_0}{4\pi}$$

Therefore we find that the angular distribution of the scattered neutrons

$$\frac{dN_n}{d\Omega(\theta)} = \frac{dN_p}{dT_p} \frac{T_0}{4\pi}$$

Since $\frac{dN_p}{dT_p} = \text{const}$; $\frac{dN_n}{d\Omega}$ is also constant. That is, the scattering of neutrons by protons is isotropic in the C.M

(9)

The differential cross section $\sigma(\Theta)$ in the CM system for the scattering into an element of solid angle $d\Omega'$ at a particular CM angle Θ is defined as

$$\sigma(\Theta) = \frac{\begin{pmatrix}\text{Number of interactions per target particle}\\ \text{that lead to scattering into } d\Omega' \text{ at the angle } \Theta\end{pmatrix}}{\begin{pmatrix}\text{Number of incident particles}\\ \text{per unit area}\end{pmatrix}}$$

If dN is the number of scattered particles into $d\Omega^*$ per unit time, then

$$\sigma(\Theta)\, d\Omega^* = \frac{dN}{I}$$

The number of particles scattered into the interval from Θ to $\Theta + d\Theta$ is proportional to

$$dN \propto \sigma(\theta) \sin\theta \, d\theta = -\sigma(\theta) \, d(\cos\theta)$$

For elastic collisions

$$T_0 = T_1 + T_2$$

and from eq. (A34)

$$\frac{T_1}{T_0} = 1 - \frac{2 m_1 m_2}{(m_1 + m_2)^2} (1 - \cos\theta)$$

or solving for $\cos\theta$

$$\cos\theta = \frac{T_{max} - 2T_2}{T_{max}}$$

where $T_{max} = \frac{4 m_1 m_2}{(m_1 + m_2)^2} T_0$ is the

maximum energy obtainable by the

recoil particle in the lab system

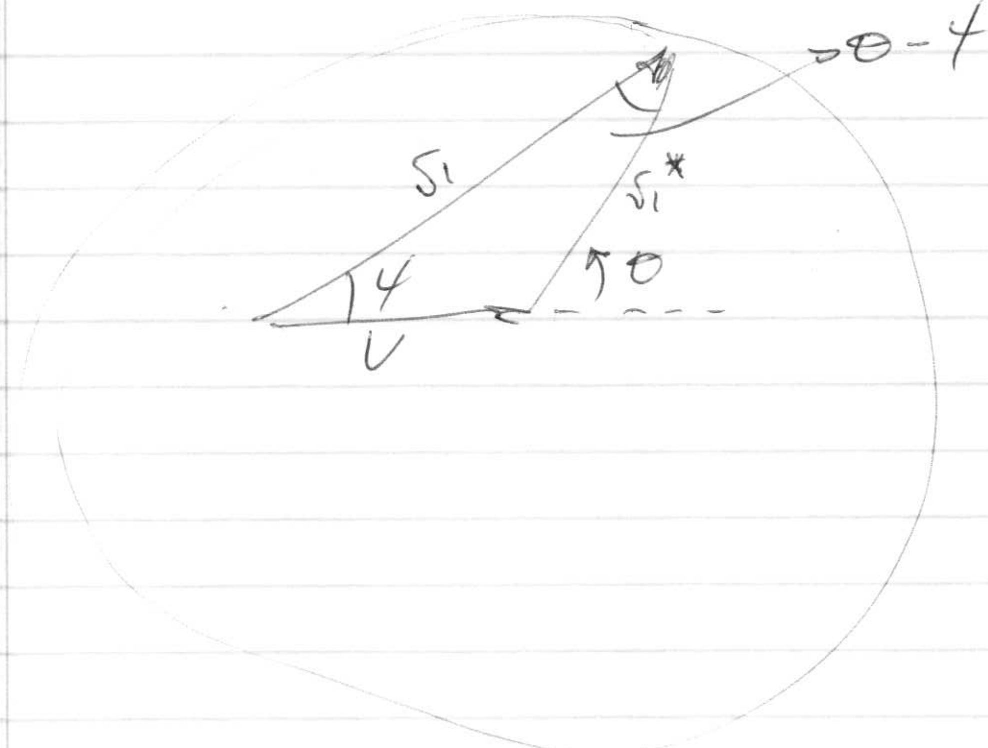
Then (1) can be re-written as

$$dN = 2\sigma(\theta) \frac{dT_2}{T_{max}}$$

and consequently we obtain the result for the energy distribution

$$\frac{dN}{dT_2} \propto \sigma(\Theta)$$

Appendix figure A.



$V < v_1^*$

Appendix:

ELASTIC COLLISIONS OF TWO PARTICLES

$m_1 =$ } mass of the { moving } particle
$m_2 =$ } mass of the { struck } particle

$u_1 =$ initial } velocity of m_1 in the LAB
$v_1 =$ final } velocity of m_1 in the LAB

$u_1^* =$ initial } velocity of m_1 in the CM
$v_1^* =$ final } velocity of m_1 in the CM

Similarly for u_2, v_2, u_2^*, and v_2^*

but $(u_2 = 0)$.

$T_0 =$ } Total initial kinetic energy in { LAB
$T_0^* =$ } Total initial kinetic energy in { CM

$T_1 =$ } Final kinetic energy of m_1 in { LAB
$T_1^* =$ } Final kinetic energy of m_1 in { CM

and similarly for T_2 and T_2^*

V = velocity of CM in the LAB

ψ = angle through which m_1 is deflected in the LAB

ξ = angle through which m_2 is deflected in the LAB

θ = angle through which m_1 and m_2 are deflected in the CM.

According to the def of CM

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = M\vec{R} \qquad (A1)$$

differentiating with respect to time

$$m_1 \vec{u}_1 + m_2 \vec{u}_2 = MV \qquad (A2)$$

Into $\vec{u}_2 = 0 \Rightarrow$

$$V = \frac{m_1 u_1}{m_1 + m_2} \qquad (A3)$$

By the same reasoning because m_2 is initially at rest

$$u_2^* = V = \frac{m_1 u_1}{m_1 + m_2} \qquad (A4)$$

Note, however, that the motion and velocities are opposite in direction and that vectorially $\vec{u}_2^* = -\vec{V}$

In the CM, the total momentum of the system is zero so

$$u_1^* = v_1^* \qquad u_2^* = v_2^* \qquad (A5)$$

Note that u_1 is the relative speed of the two particles in either the CM or LS system, $u_1 = u_1^* + u_2^*$.

Then for the final speeds we have

$$v_2^* = \frac{m_1 u_1}{m_1 + m_2} \quad (A6)$$

$$v_1^* = u_1 - u_2^* = \frac{m_2 u_1}{m_1 + m_2} \quad (A7)$$

We have from Fig A.

$$v_1^* \sin\theta = v_1 \sin\psi \quad (A8)$$

and

$$v_1^* \cos\theta + V = v_1 \cos\psi \quad (A9)$$

Dividing (A8) by (A9)

$$\tan\psi = \frac{v_1^* \sin\theta}{v_1^* \cos\theta + V} = \frac{\sin\theta}{\cos\theta + (V/v_1^*)} \quad (A10)$$

From Eq. (A3) and (A7), V/v_1^* is given by

$$\frac{V}{v_1^*} = \frac{m_1 u_1/(m_1+m_2)}{m_2 u_1/(m_1+m_2)} = \frac{m_1}{m_2} \quad (A11)$$

If we combine eq (A10) and (A11)

$$\tan\psi = \frac{\sin\theta}{\cos\theta + (m_1/m_2)} \qquad (A12)$$

we see that if $m_1 \ll m_2$, the LAB and CM scattering angles are approximately equal:

$$\psi \simeq \theta \quad \text{for } m_1 \ll m_2 \qquad (A13)$$

However, if $m_1 = m_2$

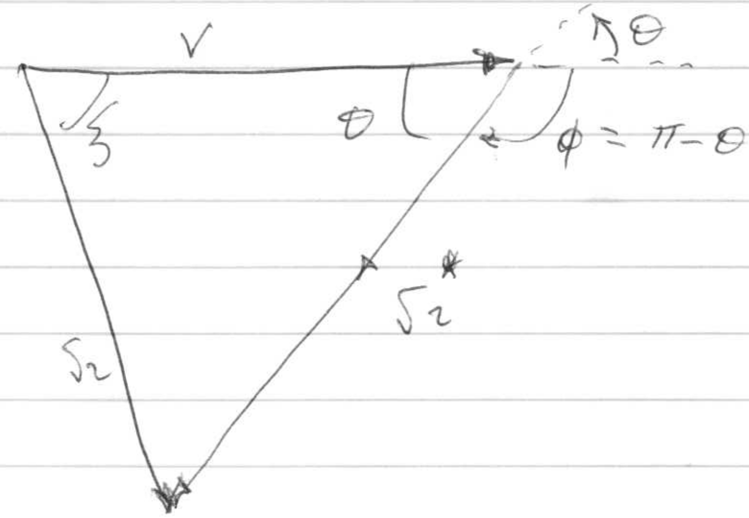
$$\tan\psi = \frac{\sin\theta}{\cos\theta + 1} = \tan\frac{\theta}{2} \qquad (A14)$$

so that

$$\psi = \frac{\theta}{2} \quad \text{for } m_1 = m_2 \qquad (A15)$$

and the LAB scattering angle is one half the CM scattering angle.

For the recoil particle we



$$v_2 \sin \xi = v_2^* \sin \theta \qquad (A16)$$

$$v_2 \cos \xi = V - v_2^* \cos \theta \qquad (A17)$$

Dividing (A16) by (A17) we have

$$\tan \xi = \frac{v_2^* \sin \theta}{V - v_2^* \cos \theta} = \frac{\sin \theta}{(V/v_2^*) - \cos \theta} \qquad (A18)$$

But according to (A4) and (A6)

V and v_2^* are equal so

$$\tan \xi = \frac{\sin \theta}{1 - \cos \theta} = \cot \frac{\theta}{2} \qquad (A19)$$

which we may write as

$$\tan \xi = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \qquad (A20)$$

$$\Rightarrow 2\xi = \pi - \theta = \phi$$

For particles with equal mass we have $m_1 = m_2$ and $\theta = 2\psi$.

Combining this result in Eq. (A20)

$$\xi + \psi = \pi/2 \qquad \text{for } m_1 = m_2 \qquad (A21)$$

Relationship involving the energies of particles may be obtained as follows.

$$T_0 = \frac{1}{2} m_1 u_1^2 \qquad (A22)$$

and, in the CM system

$$T_0^* = \frac{1}{2}\left(m_1 u_1^{*2} + m_2 u_2^{*2}\right) \qquad (A23)$$

which using (A6) and (A7) becomes

$$T_0^* = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}u_1^2 = \frac{m_2}{m_1+m_2}T_0 \qquad (A\ 23^{1/2})$$

This results shows that the initial kinetic energy in the CM system T_0' is always a fraction $\frac{m_2}{m_1+m_2} < 1$ of the initial LAB energy. For final CM energies we found

$$T_1^* = \frac{1}{2}m v_1^{*2} = \frac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\right)^2 u_1^2$$

$$= \left(\frac{m_2}{m_1+m_2}\right)^2 T_0 \qquad (A24)$$

and

$$T_2^* = \frac{1}{2}m_2 v_2^{*2} = \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\right)^2 u_1^2$$

$$= \frac{m_1 m_2}{(m_1+m_2)^2}T_0 \qquad (A25)$$

To obtain T_1 in terms of T_0, we write

$$\frac{T_1}{T_0} = \frac{\frac{1}{2} m v_1^2}{\frac{1}{2} m u_1^2} = \frac{v_1^2}{u_1^2} \qquad (A26)$$

Using the cosine law (referring to Fig A)

$$v_1^{*2} = v_1^2 + V^2 - 2 v_1 V \cos \psi \qquad (A27)$$

or

$$\frac{T_1}{T_0} = \frac{v_1^2}{u_1^2} = \frac{v_1^{*2}}{u_1^2} - \frac{V^2}{u_1^2} + \frac{2 v_1 V}{u_1^2} \cos \psi \qquad (A28)$$

From the previous definitions we have

$$\frac{v_1^*}{u_1} = \frac{m_2}{m_1 + m_2} \quad \text{and} \quad \frac{V}{u_1} = \frac{m_1}{m_1 + m_2} \qquad (A29)$$

The squares of these quantities give the desired expressions for the first two terms on the right hand side of (A28). ~~To~~ To evaluate the third term we write, using (A8)

$$\frac{2\, v_1 V}{u_1^2} \cos\psi = 2\left(v_1^* \frac{\sin\theta}{\sin\psi}\right) \frac{V}{u_1^2} \cos\psi \quad \text{(A30)}$$

The quantity $v_1^* V/u_1^2$ can be obtained from the product of the equations in (A29) and using of (A8) we have

$$\frac{\sin\theta \cos\psi}{\sin\psi} = \frac{\sin\theta}{\tan\psi} = \cos\theta + \frac{m_1}{m_2} \quad \text{(A31)}$$

So that

$$\frac{2 v_1 V}{u_1^2} \cos\psi = \frac{2 m_1 m_2}{(m_1 + m_2)} \left(\cos\theta + \frac{m_1}{m_2} \right)$$

(A32)

Substituting (A29) and (A32) into (A28) we obtain

$$\frac{T_1}{T_0} = \left(\frac{m_2}{m_1+m_2}\right)^2 - \left(\frac{m_1}{m_1+m_2}\right)^2 + \frac{2m_1m_2}{(m_1+m_2)^2}\left(\cos\theta + \frac{m_1}{m_2}\right) \quad (A33)$$

which simplifies to

$$\frac{T_1}{T_0} = 1 - \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta) \quad (A34)$$

Similarly, we can also obtain the ratio T_1/T_0 in terms of the LAB scattering angle ψ

$$\frac{T_1}{T_0} = \frac{m_1^2}{(m_1+m_2)}\left[\cos\psi \pm \sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\right]^2 \quad (A35)$$

where the plus (+) sign for the radical is to be taken unless $m_1 > m_2$ (in which case the result is double-valued.

The LAB energy of the recoil particle m_2 can be calculated from

$$\frac{T_2}{T_0} = 1 - \frac{T_1}{T_0} = \frac{4\, m_1 m_2}{(m_1+m_2)^2} \cos^2 \xi \, , \quad \xi \le \pi/2 \qquad (A36)$$

If $m_1 = m_2$, we have a simple relation

$$\frac{T_1}{T_0} = \cos^2 \psi \, , \qquad m_1 \simeq m_2 \, . \qquad (A37)$$

In addition

$$\frac{T_2}{T_0} = \sin^2 \psi \, , \qquad m_1 \simeq m_2 \qquad (A38)$$