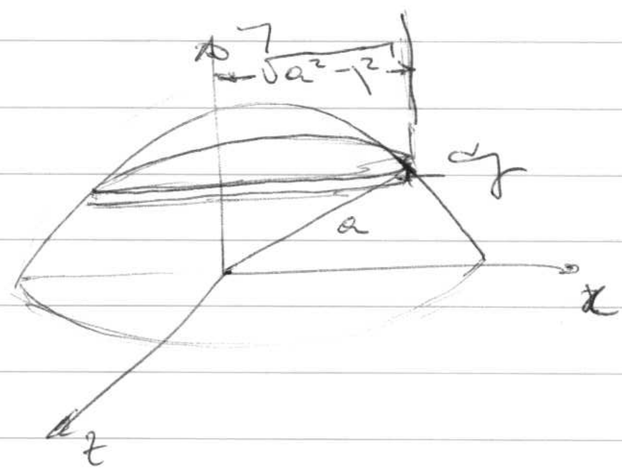
PROBLEMS # 11

1 Let the density be ρ, the hemisphere mass be M, and the radius be a

$$\rho = \frac{M}{\frac{2}{3}\pi a^2}$$



The position coordinates of R_{CM} are (x, y, z).

From symmetry $x = 0$ $z = 0$

$$X = \frac{1}{M}\int_{-a}^{a} x\, dm$$

$$Y = \frac{1}{M}\int_{-a}^{a} z\, dm$$

$$Y = \frac{1}{M}\int_0^a y\, dm$$

$$dm = \rho\, dV = \rho\pi\left(a^2 - y^2\right) dy$$

$$Y = \frac{1}{M}\int_0^a \rho\pi\, y\left(a^2 - y^2\right) dy$$

$$Y = \frac{4\pi\rho a^4}{4M} = \frac{3a}{8}$$

$$R_{cm} = \left(0,\, 3a/8,\, 0\right)$$

(2) Put the shell in the $z>0$ region with the base in the x,y plane.

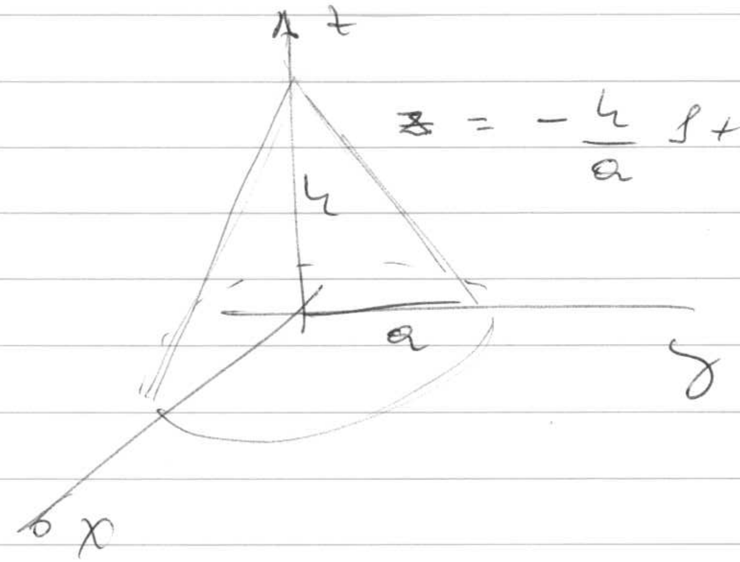
By symmetry $x_{cm} = y_{cm} = 0$

$$z = \frac{\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=r_1}^{r_2} \rho\, z\, r^2\, dr\, \sin\theta\, d\theta\, d\phi}{\int_{\phi=0}^{2\pi}\int_{0}^{\pi/2}\int_{r=r_1}^{r_2} \rho\, r^2\, dr\, \sin\theta\, d\theta\, d\phi}$$

Using $z = r\cos\theta$ and doing the integral gives

$$z_{CM} = \frac{3}{8}\,\frac{(r_2^4 - r_1^4)}{(r_2^3 - r_1^3)}$$

(3)



$$z = -\frac{h}{a}\rho + h$$

By symmetry $x_{CM} = y_{CM} = 0$

We use cylindrical coordinates

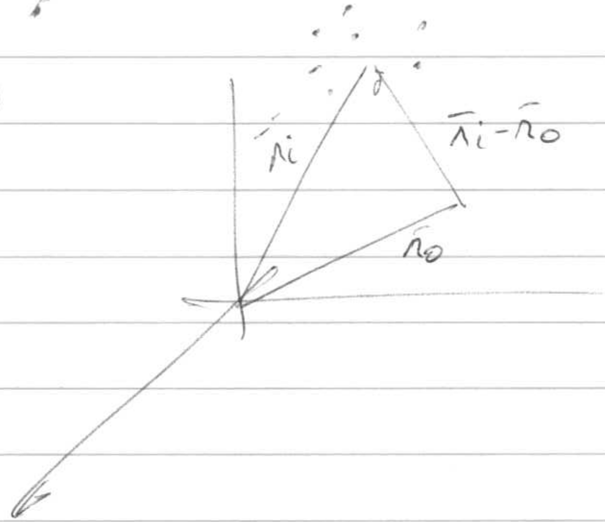
ρ_0 = mass density

$$z_{CM} = \frac{\int_0^{2\pi}\int_0^{a}\int_0^{-\frac{h}{a}\rho + h} \rho_0\, z\, \rho\, d\rho\, d\phi\, dz}{\int_0^{2\pi}\int_0^{a}\int_0^{-\frac{h}{4}\rho + h} \rho_0\, \rho\, d\rho\, d\phi\, dz} = \frac{h}{4}$$

the center-of-mass is on the axis of

the cone $\frac{3}{4}h$ from the vertex

(4)



r_i = position of the i^{th} particle

m_i = mass of the i^{th} particle

$M = \sum_i m_i$ = total mass

g = gravitational constant field

$$\tau = \sum \tau_i$$

$$= \sum (\vec{r}_i - \vec{r}_0) \times \bar{F}_i$$

$$= \sum (\bar{r}_i - r_0) \times m_i g$$

$$= \sum r_i \times m_i g - \sum r_0 \times m_i g$$

$$= \sum r_i m_i \times g - \left(\sum m_i\right) r_0 \times g$$

$$= \left(\sum m_i r_i\right) \times g - M r_0 \times g$$

Now if the total torque is zero

$$\sum m_i r_i = M r_0$$

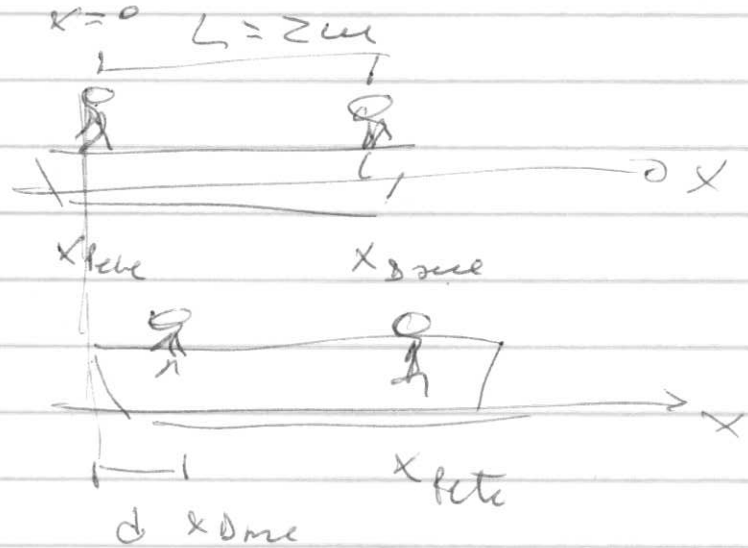
or

$$r_0 = \frac{1}{M} \sum m_i r_i$$

which is the definition of the center of mass. So $\tau = 0$ about $r_0 = r_{CM}$, or

center of gravity = center of mass

(5) Flesh out $Mx_{CM} = \sum m_i x_i$ both before and after Pete and Dave change places. The coordinate axis measures positions in the reference frame of the water



$$Mx_{CM_i} = m_{Pete}\, x_{Pete\,i} + m_{Dave}\, x_{Dave\,i} + m_{boat}\, x_{boat\,i}$$

$$Mx_{CM_f} = m_{Pete}\, x_{Pete\,f} + m_{Dave}\, x_{Dave\,f} + m_{boat}\, x_{boat\,f}$$

$$M\Delta x_{CM} = m_{Pete}\, \Delta x_{Pete} + m_{Dave}\, \Delta x_{Dave} + m_{boat}\, \Delta x_{boat}$$

$$0 = m_{Pete}(d+L) + m_{Dave}(d+2) + m_{boat}\, d$$

$$d = \frac{(m_{Dave} - m_{Pete})}{m_{Dave} + m_{Pete} + m_{boat}} \qquad L = 0.31\,m$$

(6) Let the velocity of the projectile of mass M be v. The three fragments have the following masses and velocities

$m_1 = \frac{M}{2}$, $v_1 = k_1 v$ Forward direction $k_1 > 0$

$m_2 = \frac{M}{6}$, $v_2 = -k_2 v$ opposite direction $k_2 > 0$

$m_3 = \frac{M}{3}$, $v_3 = 0$ at rest

The conservation of linear momentum and energy give

$$Mv = \frac{M}{2} k_1 v - \frac{M}{6} k_2 v \qquad (1)$$

$$E + \frac{1}{2} M v^2 = \frac{1}{2} \frac{M}{2} (k_1 v)^2 + \frac{1}{2} \frac{M}{6} (k_2 v)^2 \qquad (2)$$

From Eq. (1)

$$k_2 = 3k_1 - 6$$

which we can insert into Eq. (2)

$$5\left(\frac{1}{2}Mv^2\right) + \frac{1}{2}Mv^2 = \frac{Mv^2}{4}k_1^2 + \frac{Mv^2}{12}(3k_1 - 6)^2$$

which reduces to $k_1^2 - 3k_1 = 0$, giving the results $k_1 = 0$ and $k_1 = 3$. For $k_1 = 0$ the value of $k_2 = -6$, which is inconsistent with the value of $k_2 > 0$. For $k_1 = 3$, the value of $k_2 = 3$. The velocities become

$$v_1 = 3v$$

$$v_2 = -3v$$

$$v_3 = 0$$

(7) Since the initial kinetic energies of the two particles are equal we have

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_2 u_2^2 = \frac{1}{2} \alpha^2 m_2 u_1^2 \quad (1)$$

or

$$\frac{m_1}{m_2} = \alpha^2 \quad (2)$$

Now the kinetic energy of the system is conserved because the collision is elastic. Therefore

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = m_1 u_1^2 = \frac{1}{2} m_2 v_2^2 \quad (3)$$

Since $v_1 = 0$. Momentum is also conserved so we can write

$$m_1 u_1 + m_2 u_2 = (m_1 + \alpha m_2) u_1 = m_2 v_2 \quad (4)$$

Substituting the second equality in (4) into (3) we find

$$m_1 u_1^2 = \frac{1}{2} m_2 \left[\frac{m_1 + \alpha m_2}{m_2} \right]^2 u_1^2 \quad (5)$$

or

$$m_1 = \frac{1}{2} m_2 \left[\frac{m_1}{m_2} + \alpha \right]^2 \quad (6)$$

Using $m_1/m_2 = \alpha^2$ (6) becomes

$$2\alpha^2 = (\alpha^2 + \alpha)^2$$

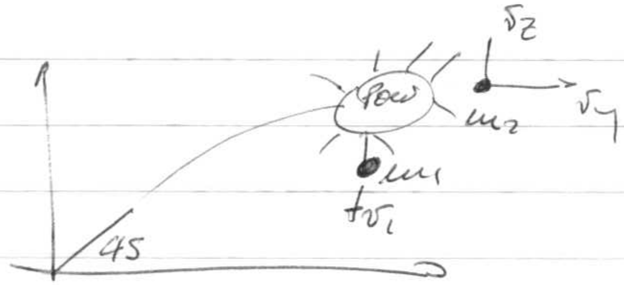
solving for α we obtain

$$\alpha_1 = -1 \pm \sqrt{2} \qquad \alpha_1^2 = 3 \mp 2\sqrt{2}$$

This gives us

$$\frac{m_1}{m_2} = 3 \pm 2\sqrt{2}, \quad \frac{u_2}{u_1} = -(1 \pm \sqrt{2}) \text{ while } \begin{cases} + : \alpha < 0 \\ - : \alpha > 0 \end{cases}$$

(8)



Let the axis be as shown with the projectile in the y-z plane. At the top just before the explosion, the velocity is in the y direction and has magnitude

$$v_{0y} = \frac{v_0}{\sqrt{2}}$$

$$v_{0y} = \frac{v_0}{\sqrt{2}} = \frac{\sqrt{\frac{2E_0}{m_1+m_2}}}{\sqrt{2}} = \sqrt{\frac{E_0}{m_1+m_2}}$$

where m_1 and m_2 are the masses of the fragments. The initial momentum is

$$P_i = (m_1+m_2)\left[0, \sqrt{\frac{E_0}{m_1+m_2}}, 0\right]$$

The final momentum is

$$P_f = P_1 + P_2$$

$$P_1 = m_1 (0, 0, v_1)$$

$$P_2 = m_2 (v_x, v_y, v_z)$$

The conservation of momentum equations are

P_x: $0 = m_2 v_x$ or $v_x = 0$

P_y: $\sqrt{E_0 (m_1 + m_2)} = m_2 v_y$

or $v_y = \frac{1}{m_2} \sqrt{E_0 (m_1 + m_2)}$

P_z: $0 = m_1 v_1 + m_2 v_z$

$$v_1 = -\frac{m_2}{m_1} v_z$$

The energy equation is

$$\frac{1}{2}(m_1 + m_2)\frac{E_0}{(m_1 + m_2)} + E_0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 (v_y^2 + v_z^2)$$

or

$$3E_0 = m_1 v_1^2 + m_2 (v_y^2 + v_z^2)$$

Substituting for v_y and v_1 gives

$$v_z = \sqrt{\frac{E_0 m_1 (2m_2 - m_1)}{m_2^2 (m_1 + m_2)}}$$

$v_1 = -\frac{m_2}{m_1} v_z$ gives

$$v_1 = -\sqrt{\frac{E_0 (2m_2 - m_1)}{m_1 (m_1 + m_2)}}$$

So m_1 travels straight down with speed $|v_1|$

v_2 travels in the y-z plane

$$v_2 = (v_y^2 + v_z^2)^{1/2} = \sqrt{\frac{E_0 (4m_1 + m_2)}{m_2 (m_1 + m_2)}}$$

$$\theta = \arctan \frac{v_z}{v_y} = \arctan \frac{\sqrt{m_1 (2m_2 - m_1)}}{(m_1 + m_2)}$$

(9) Conservation of momentum requires v_f to be in the same direction as u_1 (component of v_f $\perp$ to u_1 must be zero).

$$p_i = m_1 u_1$$

$$p_f = (m_1 + m_2)\, v_f$$

$$p_i = p_f \Rightarrow v_f = \frac{m_1}{m_1+m_2}\, u_1$$

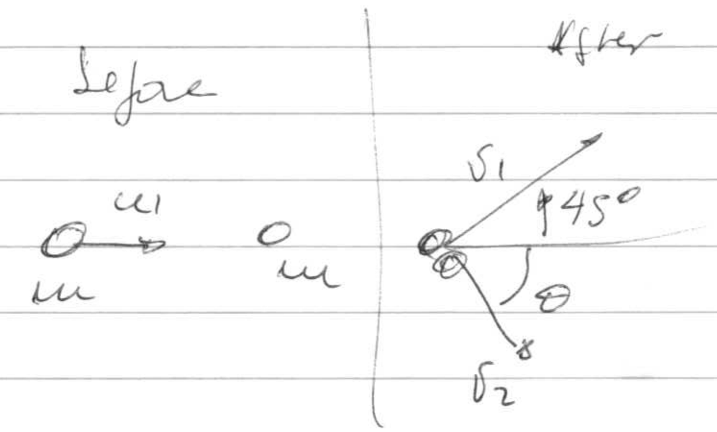
The fraction of the original kinetic energy lost is

$$\frac{K_i - K_f}{K_i} = \frac{\tfrac{1}{2} m_1 u_1^2 - \tfrac{1}{2}(m_1+m_2)\, m_1^2 u_1^2/(m_1+m_2)^2}{\tfrac{1}{2} m_1 u_1^2}$$

$$= \frac{m_1 - m_1^2/(m_1+m_2)}{m_1}$$

$$= \frac{m_2}{m_1+m_2}$$

(10)



conservation of p_x:

$$m u_1 = m v_1 \cos 45° + m v_2 \cos\theta \qquad (1)$$

conservation of p_y:

$$0 = m v_1 \sin 45° - m v_2 \sin\theta \qquad (2)$$

Conservation of energy (elastic collision)

$$\frac{1}{2} m u_1^2 = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_2^2 \qquad (3)$$

solve (1) for $\cos\theta$

$$\cos\theta = \frac{u_1 - v_1/\sqrt{2}}{v_2}$$

solve (2) for $\sin\theta$

$$\sin\theta = \frac{v_1}{\sqrt{2}\, v_2}$$

Substitute into $\cos^2\theta + \sin^2\theta = 1$, simplify

and the result is

$$u_1^2 = v_2^2 - v_1^2 + \sqrt{2}\, u_1 v_1$$

Combining this with (3) gives

$$2 v_1^2 = \sqrt{2}\, u_1 v_1$$

we know $v_1 \neq 0 \Rightarrow$

$$v_1 = u_1/\sqrt{2}$$

Substitute into (3) and the result is

$$v_2 = u_1/\sqrt{2}$$

Since $v_1 = v_2$, (2) implies that

$$\theta = 45^\circ$$

(11) Using $y = v_0 t - \frac{1}{2} g t^2$ and $v = v_0 - gt$,

we can get the velocities before and after the collision

Before $u_1 = -g t_1$ where $h_1 = \frac{1}{2} g t_1^2$

So $u_1 = -g \sqrt{\frac{2h_1}{g}} = -\sqrt{2gh_1}$

After $0 = v_0 - g t_2$ or $t_2 = v_0/g$

$$h_2 = v_0 t_2 - \tfrac{1}{2} g t_2^2$$

$$= \frac{v_0^2}{g} - \frac{1}{2}\frac{v_0^2}{g} \quad \text{or} \quad v_0 = \sqrt{2gh_2}$$

So $v_1 = \sqrt{2gh_2}$

Thus

$$e = \frac{|v_2 - v_1|}{|u_2 - u_1|} = \frac{\sqrt{2gh_2}}{\sqrt{2gh_1}}$$

$$e = \sqrt{\frac{h_2}{h_1}}$$

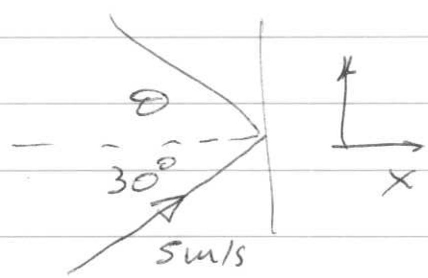
$$T_{lost} = T_i - T_f$$

$$\text{fraction lost} = \frac{T_i - T_f}{T_i}$$

$$= \frac{u_1^2 - v_1^2}{u_1^2} = \frac{h_1 - h_2}{h_1} = 1 - \frac{h_2}{h_1}$$

$$\frac{T_i - T_f}{T_i} = 1 - e^2$$

(12)



The velocity component in the y direction is unchanged

$$v_y = u_y = 5\,\frac{m}{s}\,\sin 30° = 2.5\ m/s$$

For the x component we have

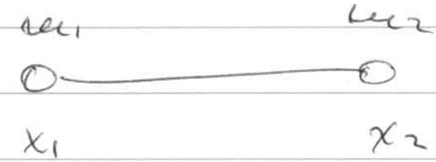
$$0.8 = \frac{|v_x|}{|u_x|} = \frac{v_x}{\left(5\,\frac{m}{s}\right)\cos 30^\circ} = \frac{v_x}{\frac{\sqrt{3}}{2}\,5\ m/s}$$

$$|v_x| = 2\sqrt{3}\ m/s$$

$$v_f = \frac{1}{2}\sqrt{73} \simeq 4.3\ m/s$$

$$\theta = \mathrm{ARCTAN}\ \frac{2.5}{2\sqrt{3}} \simeq 36^\circ$$

(13)



$$r = |\vec{x}_2 - \vec{x}_1|$$

when two particles are initially at rest separated by a distance r_0, the system has the total energy

$$E_0 = -G\frac{m_1 m_2}{r_0} \qquad (1)$$

The coordinates of the particles, x_1 and x_2, are measured from the position of the center of mass. At any time the total energy is

$$E = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{G m_1 m_2}{r} \qquad (2)$$

and the linear momentum, at any time, is

$$p = m_1 \dot{x}_1 + m_2 \dot{x}_2 = 0 \qquad (3)$$

From the conservation of energy we have

$E = E_0$, or

$$-G \frac{m_1 m_2}{r_e} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 x_2^2 - \frac{G m_1 m_2}{r}$$

(4)

Using (3) in (4), we find

$$\dot{x}_1 = v_1 = m_2 \sqrt{\frac{2G}{M}\left[\frac{1}{r} - \frac{1}{r_0}\right]}$$

$$\dot{x}_2 = v_2 = -m_1 \sqrt{\frac{2G}{M}\left[\frac{1}{r} - \frac{1}{r_0}\right]}$$

(14)

(a) $v = v_0 + u \ln \frac{m_0}{m}$

Assuming $v_0 = 0$, we have

$$v = \left[100 \frac{m}{s}\right] \ln \frac{100}{98}$$

$v = 2.02$ m/s ; so, he runs out of gas

(b) Relative to Sojeet original frame of reference we have

Before throwing tank

98 kg → 2.02 m/s

After throwing tank we want Sujeet velocity to be slightly greater than 3 m/s

(so he will catch up the orbiter)

v ← 8 kg   90 kg → 3 m/s

Conservation of momentum gives

$$98\text{ kg }(2.02\text{ m/s}) = (90\text{ kg})(3\text{ m/s}) - (8\text{ kg})v$$

$$v = 9\text{ m/s}$$

(This velocity is relative to Sujeet reference frame, i.e., before he gives his pressurized tank.) Since Sujeet is travelling towards the orbiter at 2.02 m/s, he must throw the tank at $v = 9\text{ m/s} + 2.02\text{ m/s}$

$$v = 11\text{ m/s}$$