PROBLEMS # 10

(1)

The kinetic and potential energies are

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \text{and} \quad U = mgz$$

Therefore

$$\mathcal{L} = T - U = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

The 3 Lagrange equations are

$$0 = m\ddot{x} \quad , \quad 0 = m\ddot{y} \; , \text{ and } \; -mg = m\ddot{z}$$

which are just the 3 components of the equation $\vec{F} = m\vec{a}$ for a projectile with $\vec{F} = m\vec{g}$.

(2) With $F = -kx$, the potential energy is $U = \frac{1}{2} k x^2$ and the Lagrangian

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

The Lagrange equation is $-kx = m\ddot{x}$ and its solution is $x = A\cos(\omega t - \delta)$ where $\omega = \sqrt{k/m}$ and A and δ are arbitrary constants.

(3)

$$\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} k(x^2+y^2)$$

and the two Lagrange equations are

$$-kx = m\ddot{x}$$

$$-ky = m\ddot{y}$$

In general the solution, x and y oscillate with the same angular frequency $\omega = \sqrt{k/m}$ and the point (x, y) moves around an ellipse

(4) The requested equations are just the standard equations for 2 dimensional polar coordinates, with the radius fixed at that of the hoop, $r = R$, namely

$$x = R \cos\phi$$

$$y = R \sin\phi$$

in the other direction

$$\phi = \arctan(y/x)$$

with ϕ chosen to be in the right quadrant

(5)

$$x = \rho \cos\phi$$

$$y = \rho \sin\phi$$

$$z = \rho / \tan\alpha$$

and in the other direction

$$\rho = \sqrt{x^2+y^2}$$

or $\rho = z \tan \alpha$

and $\phi = \arctan(y/x)$ with ϕ chosen

to lie in the right quadrant

(e)

The kinetic energy of rotation of

the pulley is $\frac{1}{2} I \omega^2 = \frac{1}{2} \frac{I \dot{x}^2}{R^2}$

since $\omega = \dot{x}/R$. Therefore the total

kinetic energy is

$$T = \frac{1}{2}\left(m_1 + m_2 + I/R^2\right)\dot{x}^2$$

while the potential energy is

$$U = -(m_1 - m_2) g x \qquad (m_1 > m_2)$$

Thus the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}\left(m_1 + m_2 + I/R^2\right)\dot{x}^2 + \left(m_1 - m_2\right)gx$$

and the Lagrange equation is

$$\left(m_1 - m_2\right)g = \left(m_1 + m_2 + I/R^2\right)\ddot{x}$$

That is $\ddot{x} = \dfrac{(m_1 - m_2)g}{(m_1 + m_2 + I R^2)}$

(7) We must first write down the Lagrangian in an inertial frame, for which the natural choice is a frame fixed to the earth, relative to which the elevator is accelerating upward. The point of support in the elevator's ceiling has velocity $V = (0, at)$. (if we measured x horizontally and y vertically up) and position $(0, \frac{1}{2}at^2)$. The bob's velocity relative to the elevator is $\vec{v}_{rel} = (l\dot{\phi}\cos\phi, \; l\dot{\phi}\sin\phi)$. Thus its velocity relative to the ground is

$$\vec{v} = \vec{V} + \vec{v}_{rel} = (l\dot{\phi}\cos\phi, \; at + l\dot{\phi}\sin\phi)$$

The bob's height above the ground is

$$y = \frac{1}{2} a t^2 - l \cos\phi$$

After a little algebra the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} m v^2 - m g y$$

$$= \frac{1}{2} m \left(a^2 t^2 + 2 a t l \dot{\phi} \sin\phi + l^2 \dot{\phi}^2 \right) - m g \left(\frac{1}{2} a t^2 - l \cos\phi \right)$$

The Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \Rightarrow$$

$$m a t l \dot{\phi} \cos\phi - m g l \sin\phi = \frac{d}{dt} \left(m l^2 \dot{\phi} + m a t l \sin\phi \right)$$

$$= m l^2 \ddot{\phi} + m a t l \dot{\phi} \cos\phi + m a l \sin\phi$$

Making a couple of cancellations and re-arranging, we arrive at the equation

$$l\ddot{\phi} = -(g+a)\sin\phi$$

which is the equation for a normal (non-accelerating) pendulum, except that g has been replaced by $(g+a)$

(8) If $\vec{F} = k r^n \hat{r}$, then $U(r) = -\int_{r_0}^{r} \vec{F}(r')\cdot \vec{dr}'$

$$U(r) = -\int_{r_0}^{r} k\, r'^n\, dr' = -\frac{k\, r^{n+1}}{n+1} + C$$

The potential energy is then

$$U(r) = -\frac{k\, r^{n+1}}{n+1}$$

where we have chosen $C = 0$.

(a) The small cart's kinetic energy is

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x} + \dot{X})^2 = \frac{1}{2} m (\dot{x} - A\omega \sin \omega t)^2$$

and $U = \frac{1}{2} k x^2$.

Thus $\frac{\partial \mathcal{L}}{\partial \dot{x}} = m(\dot{x} - A\omega \sin \omega t)$

and Lagrange's equation reads

$$-kx = m\ddot{x} - m A\omega^2 \cos \omega t$$

or

$$\ddot{x} + \omega_0^2 x = B \cos \omega t$$

where we have replaced k/m by ω_0^2 and rename $A\omega^2$ as B.

(10) The potential energy is

$$U = mgz = mgk\rho^2$$

The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\dot{\rho}^2 + (\rho\dot{\phi})^2 + \dot{z}^2\right]$$

$$= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\omega^2 + 4k^2\rho^2\dot{\rho}^2\right)$$

Therefore the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\omega^2 + 4k^2\rho^2\dot{\rho}^2\right) - mgk\rho^2 \quad (1)$$

and the equation of motion is (after a bit of algebra)

$$(1+4k^2\rho^2)\ddot{\rho} + \cancel{4}k^2\rho\dot{\rho}^2 = (\omega^2 - 2gk)\rho. \quad (2)$$

A position ρ_0 is an equilibrium point if placing the bead at ρ_0 with $\dot{\rho} = 0$

ensures that $\ddot{\rho}=0$ (guaranteeing that $\dot{\rho}$ remains zero and ρ constant).

According to Eq. (2) this will be true if and only if

$$(\omega^2 - 2gk)\rho = 0 \qquad (3)$$

This can be satisfied in two ways: First, the system is in equilibrium if $\rho = 0$, that is, if the bead is exactly at the bottom of the wire. To decide whether this equilibrium is stable, we have only to imagine pulling the bead a small distance to one side. With ρ and $\dot{\rho}$ very small eq (2) implies that

$$\ddot{s} \approx (\omega^2 - 2gk)\, s$$

If $\omega^2 < 2gk$, the term in parenthesis is negative, and the bead accelerates back to the bottom, so the equilibrium is stable. If $\omega^2 > 2gk$, the term in parenthesis is positive, and the bead accelerates away from the bottom, so the equilibrium is unstable.

If $\omega^2 = 2gk$, the condition (3) is satisfied for any value of s. That is with $\omega = \sqrt{2gk}$, the bead will be in equilibrium anywhere on the wire

To investigate the stability of this equilibrium, we imagine giving the bead a small nudge. From eq (2), with the right side fixed to zero, we see that $\ddot{\xi}$ is always negative for any possible ξ. ($\xi > 0$). Thus if we nudge the bead outward, its negative acceleration will slow it down, though not actually stop it. If we nudge it inward it will actually speed up. Either way it will not return to its original position.

(11) (a) The position and hence velocity of mass M are

$$\vec{r}_M = (x + L\sin\phi,\ L\cos\phi)$$

$$\vec{v}_M = (\dot{x}_M + \dot{y}_M) = (\dot{x} + L\dot{\phi}\cos\phi,\ -L\dot{\phi}\sin\phi)$$

Therefore the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left(m v_m^2 + M v_M^2\right) + M g y_M - \frac{1}{2} k x^2$$

$$= \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}M\left(L^2\dot{\phi}^2 + 2\dot{x}L\dot{\phi}\cos\phi\right) + MgL\cos\phi - \frac{1}{2}kx^2$$

and the two Lagrange equations are (after a little tidying up)

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$(m+M)\ddot{x} + ML\ddot{\phi}\cos\phi - ML\dot{\phi}^2\sin\phi = -kx$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$L\ddot{\phi} + \ddot{x}\cos\phi = -g\sin\phi$$

(b) If ϕ remains small, we can write $\cos\phi \simeq 1$ and $\sin\phi \simeq \phi$ and ignore powers of ϕ or $\dot{\phi}$ higher than second to give

$$(m+M)\ddot{x} + ML\ddot{\phi} = -kx$$

and

$$L\ddot{\phi} + \ddot{x} = -g\phi$$

(12) The Lagrangian of the particle subject to a gravitational force is written in terms of the cylindrical coordinates as

$$L = T - U = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz \quad (1)$$

From the constraint $r^2 = 4az$, we have

$$\dot{z} = \frac{r\dot{r}}{2a} \qquad (2)$$

Therefore, (1) becomes

$$L = \frac{1}{2} m \left[\left(1 + \frac{r^2}{4a^2}\right) \dot{r}^2 + r^2 \dot{\theta}^2 \right] - \frac{mg}{4a} r^2$$

(3)

Lagrange equation for θ is

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -\frac{d}{dt}\left(m r^2 \dot{\theta}\right) = 0 \qquad (4)$$

This equation shows that the angular momentum of the system is conserved, as expected:

$$m r^2 \dot{\theta} = l = \text{constant} \qquad (5)$$

Lagrange equation for r is

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} =$$

$$\frac{m}{4a^2} r\dot{r}^2 + m r\dot{\theta}^2 - \frac{mg r}{2a} - \frac{d}{dt}\left[m\left(1+\frac{r^2}{4a^2}\right)\dot{r}\right] = 0$$

(6)

from which

$$\frac{m}{4a^2} r\dot{r}^2 + m r\dot{\theta}^2 - \frac{mg}{2a} r - m\left[1+\frac{r^2}{4a^2}\right]\ddot{r}$$

$$- \frac{m}{2a^2} r\dot{r}^2 = 0$$

(7)

After rearranging, this equation becomes

$$m\left[1+\frac{r^2}{4a^2}\right]\ddot{r} + \frac{m}{4a^2} r\dot{r}^2 + \frac{mg}{2a} r - \frac{\ell^2}{m}\frac{1}{r^3} = 0$$

(8)

For a circular orbit, we must have

$\dot{r} = \ddot{r} = 0$ or $r = \rho$ = constant

$$\frac{mg\rho}{2a} = \frac{l^2}{m\rho^3} \qquad (9)$$

or

$$l^2 = \frac{m^2 g}{2a}\rho^4 \qquad (10)$$

Equating this $l^2 = m^2\rho^4\dot{\theta}^2$, we have

$$m^2\rho^4\dot{\theta}^2 = \frac{m^2 g}{2a}\rho^4 \qquad (11)$$

or

$$\dot{\theta}^2 = \frac{g}{2a} \qquad (12)$$

Applying a perturbation to the circular orbit we can write

$$r \to \rho + x \quad \text{where} \quad \frac{x}{\rho} \ll 1$$

This causes the following changes

$$r^2 \to \rho^2 + 2\rho x$$

$$\frac{1}{r^3} \to \frac{1 - 3x/\rho}{\rho^3}$$

$$\dot{r} \to \dot{x}$$

$$\ddot{r} \to \ddot{x}$$

from which we have

$r\dot{r}^2 \to (\rho + x)\dot{x}^2 \simeq 0$, in lowest order

$r^2\ddot{r} \to (\rho^2 + 2\rho x)\ddot{x} \simeq \rho^2\ddot{x}$, in lowest order

Thus Eq (8) becomes

$$m\left[1 + \frac{1}{4a^2}\rho^2\right]\ddot{x} + \frac{mg}{2a}(\rho + x) - \frac{\ell^2}{m\rho^3}\left(1 - \frac{3x}{\rho}\right) = 0 \quad (13)$$

But $$\frac{mg\rho}{2a} = \frac{\ell^2}{m\rho^3} \quad (14)$$

Thus eq.(13) becomes

$$m\left[1+\frac{\rho^2}{4a^2}\right]\ddot{x}+\frac{mg}{2a}x+\frac{3\ell^2}{m\rho^4}x=0 \tag{15}$$

Substituting (14) into (15) we find

$$m\left[1+\frac{\rho^2}{4a^2}\right]\ddot{x}+\frac{2mg}{a}x=0$$

or

$$\ddot{x}+\frac{2g}{a+\frac{\rho^2}{4a}}x=0$$

Therefore the frequency of small oscillations is

$$\omega=\sqrt{\frac{2g}{a+z_0}}$$

where $z_0=\rho^2/4a$