Group Problems #9 - Solutions

Monday, September 12

Problem 1 *Relativistic kinetic energy and momentum*

The total energy of a particle with speed u is given by γmc^2, where $\gamma = 1/\sqrt{1 - u^2/c^2}$, and m is the particle's mass. This total energy is the sum of the particle's kinetic energy and the rest energy, mc^2. The relativistic 3-momentum of the particle is γmu.

(a) Write a simple expression for the kinetic energy of the particle in terms of the given quantities.

The problems states that $E = \gamma mc^2 = K + mc^2$, where K is the kinetic energy. Solving for K gives $K = (\gamma - 1)mc^2$.

(b) How fast does the particle need to move for its kinetic energy to account for 2/3 of the total energy?

The problem states that $K = (\gamma - 1)mc^2 = 2/3\gamma mc^2 \longrightarrow \gamma = 3$. So we can solve for β:

$$\frac{1}{\sqrt{1-\beta^2}} = 3 \implies 1 - \beta^2 = \frac{1}{9} \tag{1}$$

$$\implies \beta = \frac{\sqrt{8}}{3}. \tag{2}$$

So the particle must travel at $u = \sqrt{8}c/3$ for its kinetic energy to account for 2/3 of the total energy.

(c) What is the momentum of this particle?

The momentum of the particle is $p = \gamma mu = 3m\sqrt{8}c/3 = \sqrt{8}mc$. Now notice that if we know only the total energy of the particle $E = \gamma mc^2$ and its momentum $p = \gamma mu$, we can readily find its speed:

$$\frac{pc}{E} = \frac{\gamma muc}{\gamma mc^2} = \frac{u}{c} = \beta. \tag{3}$$

In this case, we have $\beta = (\sqrt{8}mc^2)/(3mc^2) = \sqrt{8}/3$ as we found above.

Problem 2 *Dot product of 4-vectors*

Show that the dot product of two 4-vectors is a scalar. That is, show that for any two 4-vectors, $\vec{a}$ and $\vec{b}$, their dot product in one frame S is equal to their dot product in another frame S' moving with respect to S: $\vec{a} \cdot \vec{b} = \vec{a'} \cdot \vec{b'}$.
For simplicity, let's assume that the S'-frame moves with velocity v in the $+x$-direction. So, the task is to compare the dot products before and after $\vec{a}$ and $\vec{b}$ have undergone a Lorentz transformation. The product in the S-frame is:

$$\vec{a} \cdot \vec{b} = a_0\, b_0 - a_1\, b_1 - a_2\, b_2 - a_3\, b_3. \tag{4}$$

The Lorentz transformation matrix for both $\vec{a}$ and $\vec{b}$ is:

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

which gives for the components of $\vec{a'}$: $a'_0 = \gamma a_0 - \beta\gamma a_1$, $a'_1 = -\beta\gamma a_0 + \gamma a_1$, $a'_2 = a_2$, and $a'_3 = a_3$. Similarly for $\vec{b'}$: $b'_0 = \gamma b_0 - \beta\gamma b_1$, $b'_1 = -\beta\gamma b_0 + \gamma b_1$, $b'_2 = b_2$, and $b'_3 = b_3$. So $\vec{a'} \cdot \vec{b'} = (\gamma a_0 - \beta\gamma a_1)(\gamma b_0 - \beta\gamma b_1) - (-\beta\gamma a_0 + \gamma a_1)(-\beta\gamma b_0 + \gamma b_1) - a_2\, b_2 - a_3\, b_3$. Multiplying, gathering, and canceling terms then gives $\vec{a'} \cdot \vec{b'} = a_0 b_0 \gamma^2 (1 - \beta^2) + a_1 b_1 \gamma^2 (\beta^2 - 1) - a_2\, b_2 - a_3\, b_3$. Now we recognize that $1 - \beta^2 = 1/\gamma^2$, so finally we have $\vec{a'} \cdot \vec{b'} = a_0\, b_0 - a_1\, b_1 - a_2\, b_2 - a_3\, b_3 = \vec{a} \cdot \vec{b}$.

Problem 3 *Magnitude of the 4-velocity*

Apply the formula for the magnitude of a 4-vector to the general 4-velocity ($\vec{v} = \gamma c, \gamma v_x, \gamma v_y, \gamma v_z$) to show that its magnitude is indeed c.
The 4-velocity is defined as $\vec{v} = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z)$. The inner product of $\vec{v}$ with itself is $\vec{v} \cdot \vec{v} = |\vec{v}|^2 = \gamma^2\, c^2 - \gamma^2\, v_x^2 - \gamma^2\, v_y^2 - \gamma^2\, v_z^2$. Now $v_x^2 + v_y^2 + v_z^2 = v^2$, so $\vec{v} \cdot \vec{v} = \gamma^2 (c^2 - v^2) = \gamma^2\, c^2 (1 - v^2/c^2) = \gamma^2 c^2/\gamma^2 = c^2$, and we see finally that $|\vec{v}| = c$.