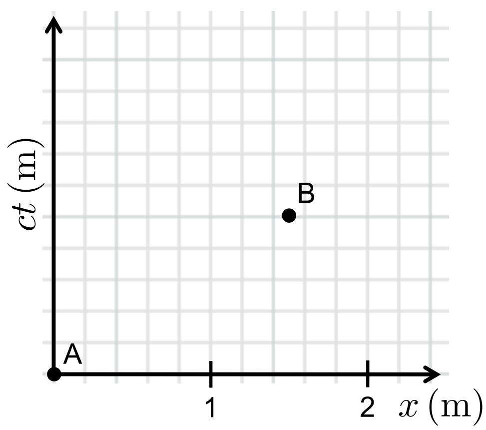
Group Problems #8 - Solutions

Friday, September 9

Problem 1 *Proper length*

Consider two events A$(ct, x) = (0, 0)$ and B(1 m, 3/2 m) in frame $S(ct, x)$.

(a) Plot the events in the $S(ct, x)$ frame.



The time interval between A and B in frame S is $c\Delta t = 1 - 0 = 1$ m and the space interval is $\Delta x = 3/2 - 0 = 3/2$ m. Thus *the invariant* interval is $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 = 1 - 9/4 = -5/4$ m^2. Since $(\Delta s)^2 < 0$, A and B are separated by a *space-like* interval and thus we can, in principle, find a frame in which the two events happen at the same time; that is, we can find the *proper distance* between them (see parts b & c).

(b) Find the velocity of the frame $S'(ct', x')$ in which the two events happen at the same time.

There are a number of ways to go about this, including using space-time diagrams. However, the most straightforward is to use the Lorentz transformation for the time coordinate: $c\Delta t' = \gamma(c\Delta t) - \beta\gamma(\Delta x)$. (This follows from the first row of the Lorentz Transformation matrix.) The problem asks you to find a frame in which the two events happen at the same time, so $\Delta t' = 0$. Thus, we have:

$$\begin{aligned} c\Delta t' = 0 &= \gamma(c\Delta t) - \beta\gamma(\Delta x) && (1)\\ &= \gamma(1\,\mathrm{m}) - \beta\gamma(3/2\,\mathrm{m}) && (2)\\ \Longrightarrow (3/2)\beta\gamma = \gamma, & && (3) \end{aligned}$$

and thus we see that $\beta = 2/3$, or $v = 2/3c$. So the two events are simultaneous in a reference frame which moves with $v = 2c/3$ in the $+x$ direction.

(c) What is the proper distance $\Delta x'$ between A and B?

The proper distance is defined as the space interval between two events that happen at the same time in a particular reference frame. There are several ways to go about finding the proper distance between events A and B, including using a Lorentz Transformation to find $\Delta x'$ with $\beta = 2/3$ as we calculated above. A more general and simple way is to use the invariant interval:

$$\begin{aligned} (\Delta s)^2 = (\Delta s')^2 \Longrightarrow (c\Delta t)^2 - (\Delta x)^2 &= (c\Delta t')^2 - (\Delta x')^2 && (4)\\ (1\,\mathrm{m})^2 - (3/2\,\mathrm{m})^2 &= (0)^2 - (\Delta x')^2 && (5)\\ -5/4\,\mathrm{m}^2 &= -(\Delta x')^2, && (6) \end{aligned}$$

and we see that $\Delta x' = \sqrt{5}/2$ m. This is the proper distance between events A and B, and as such it is the longest possible space interval between the two events.

Problem 2 *Doppler shift*

The [O II] emission line with rest-frame wavelength $\lambda_0 = 3727$ angstroms (Å) is observed in a distant galaxy to be at $\lambda = 9500$ Å. The redshift z is the *fractional change* in the observed wavelength compared to the rest-frame wavelength.

(a) What is the redshift z of the galaxy?

Here, we simply need to interpret the definition of z given above:

$$z = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{9500 - 3727}{3727} = 1.55. \quad (7)$$

(b) What is the recession speed β of the galaxy?

The Doppler shift is defined in terms of frequency ν, so we first need to convert light wavelength to frequency. For any traveling wave, the frequency is the number of cycles (crests or valleys of the wave) per second that pass a particular point in space - it is the inverse of the wave's period. Thus the frequency is $\nu = c/\lambda$, where c is the speed of the wave (the speed of light in this case) and λ is its wavelength. Using the Doppler shift expression we have:

$$\frac{\nu_{\text{obs}}}{\nu_{\text{source}}} = \frac{\sqrt{1-\beta^2}}{1+\beta\cos\theta} \tag{8}$$

$$= \frac{c\,\lambda_{\text{source}}}{c\,\lambda_{\text{obs}}} = \frac{\lambda_0}{\lambda}. \tag{9}$$

The problem asks for the "recession speed", implying that $\theta = 0$, where θ is the angle between the object's velocity vector and the line of sight between the observer and the object. Thus, we can solve for β:

$$\theta = 0 \implies \frac{\lambda_0}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}} \tag{10}$$

$$\implies \left(\frac{\lambda_0}{\lambda}\right)^2 = \frac{1-\beta}{1+\beta} \tag{11}$$

$$\implies \beta = \frac{1-\left(\frac{\lambda_0}{\lambda}\right)^2}{1+\left(\frac{\lambda_0}{\lambda}\right)^2}. \tag{12}$$

With $\lambda_0/\lambda = 3727/9500$, we have $\beta = 0.73$, or $v = 0.73\,c$.

Problem 3 *Anomalous observation*

A space probe is launched from Earth at a speed of $v = 0.8c$. On board the probe, there is a powerful beacon that emits light at a wavelength $\lambda = 500$ nm (in its rest frame). Many years after launching, NASA scientists locate the probe with a powerful telescope, and they measure the light from the beacon to have a wavelength of $\lambda = 500$ nm (in their frame). Is this possible? If we assume that the probe is still moving at $v = 0.8c$ relative to Earth, what is the explanation for this observation?
Mathematically, this problem states that $\nu_{\text{obs}}/\nu_{\text{source}} = \lambda_{\text{source}}/\lambda_{\text{obs}} = 1$. So we can use the Doppler shift equation to solve for the relationship between β and θ:

$$\frac{\lambda_{\text{source}}}{\lambda_{\text{obs}}} = \frac{\sqrt{1-\beta^2}}{1+\beta\cos\theta}. \tag{13}$$

Before we launch into the mathematical analysis, let's think about what we expect. The angle θ is defined relative to the line of sight between the observer and the object being measured, so $\theta = 0$ corresponds to the object moving directly away from the observer along the line of sight, $\theta = \pi$ corresponds to the object moving directly toward the observer, and $\theta = \pi/2$ corresponds to the object moving perpendicular to the line of sight. We know (from intuition and the equation above) that there will be a shift toward longer observed wavelengths (redshift) when $\theta = 0$ and shorter observed wavelengths (blueshift) when $\theta = \pi$. We also know that there is a (small) redshift when $\theta = \pi/2$. So somewhere between $\theta = \pi/2$ and π we must have a situation where $\lambda_{\text{source}}/\lambda_{\text{obs}} = 1$, that is where there is no shift at all between the observed and source wavelengths. Physically, this results from the fact that the Doppler shift arises from two distinct effects: time dilation, and the motion of the object relative to the observer (see Lecture #8). When the distance between the object and observer is decreasing, these two effects act in opposition, and thus under the right conditions (particular relationship between θ and β) they can exactly balance. Let's now find that relationship, first in a general way, and then for this particular problem.

Using the equation above with $\lambda_{\text{obs}} = \lambda_{\text{source}}$ gives:

$$\frac{\lambda_{\text{source}}}{\lambda_{\text{obs}}} = 1 = \frac{\sqrt{1-\beta^2}}{1+\beta\cos\theta} \implies 1+\beta\cos\theta = \sqrt{1-\beta^2} \tag{14}$$

$$\implies \cos\theta = \frac{1}{\beta}\left(\sqrt{1-\beta^2}-1\right) \tag{15}$$

The first thing to notice is that since $0 \leq \beta \leq 1$, the value of the square-root term will always be ≤ 1. Thus the right side of Eq. (15) will be negative. This matches with our logic above since $\cos\theta < 0$ for $\pi/2 < \theta \leq \pi$.

The second thing to notice is that in the limit of very small probe velocity ($\beta \to 0$), we can use a Taylor expansion for the term $\sqrt{1-\beta^2}$ in Eq. (15):

$$\beta \to 0 \Longrightarrow \cos\theta \approx \frac{1}{\beta}\left(1 - \frac{1}{2}\beta^2 - 1\right) = -\frac{\beta}{2}. \tag{16}$$

Thus in the limit $\beta \to 0$, then $\cos\theta \to 0$, and the object must travel nearly perpendicular to the line of site ($\theta \approx \pi/2$). In contrast, when $\beta \to 1$, then Eq. (15) shows that $\cos\theta \to -1$, and we see that the object must travel nearly directly toward the observer ($\theta \approx \pi$).

Now let's consider the specific case outlined in this problem, namely $\beta = 4/5$. Inserting this for β in Eq. (15) gives: $\cos\theta = 5/4(3/5-1) = 3/4 - 5/4 = -1/2$, so $\theta = \cos^{-1}(-1/2) = 120°$.