Group Problems #7 - Solutions

Wednesday, September 7

Problem 1 *The invariant interval*

Use the Lorentz transformations of Δt and Δx to show that $(\Delta s)^2$ is invariant.
For frame S' moving in the $+x$ direction with velocity v relative to frame S, the Lorentz transformation matrix is given by:

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

where $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$, and c is the speed of light in vacuum. Thus, we have $c\Delta t' = \gamma(c\Delta t) - \beta\gamma\Delta x$, and $\Delta x' = -\beta\gamma(c\Delta t) + \gamma\Delta x$. Squaring both of these equations gives:

$$(c\Delta t')^2 = [\gamma c\Delta t - \beta\gamma\Delta x]^2 \tag{2}$$
$$= \left[(\gamma c\Delta t)^2 + (\beta\gamma\Delta x)^2 - 2(\beta c\gamma^2\Delta x\Delta t)\right], \text{ and} \tag{3}$$
$$(\Delta x')^2 = \left[(\beta\gamma c\Delta t)^2 + (\gamma\Delta x)^2 - 2(\beta c\gamma^2\Delta x\Delta t\right]. \tag{4}$$

Now, the invariant interval in the S' frame is defined as $(\Delta s')^2 = (c\Delta t')^2 - (\Delta x')^2$, so

$$(\Delta s')^2 = (c\Delta t)^2\gamma^2\left[1-\beta^2\right] + (\Delta x)^2\gamma^2\left[\beta^2 - 1\right]. \tag{5}$$

where the cross terms (proportional to $\Delta x\Delta t$) have cancelled. Using the definition for γ, we see that $1-\beta^2 = 1/\gamma^2$, so the equation above reduces to:

$$(\Delta s')^2 = (c\Delta t)^2\gamma^2[1/\gamma^2] + (\Delta x)^2\gamma^2[-1/\gamma^2] \tag{6}$$
$$= (c\Delta t)^2 - (\Delta x)^2 = (\Delta s)^2. \tag{7}$$

So ultimately we see that $(\Delta s')^2 = (\Delta s)^2$. This proves in a general way that the invariant interval is, in fact, invariant!

Problem 2 *Lorentz transformation*

In Group Problems #6, a rocket ship traveling at $v = 0.8c$ in the $+x$-direction passes the Earth at noon (event A), then sends a radio signal back to Earth when it passes a space station (event B). The Earth receives the signal (event C) and relays it back toward the rocket ship, which later receives it at event D. In Earth's frame, we found the coordinates (ct, x) of these four events to be: A(0, 0); B(5/6, 2/3); C(1.5, 0); and D(7.5, 6), where all coordinates are in units of $c \cdot h$. Use a Lorentz transformation to find the coordinates of these events in the rocket ship's frame. Do they agree with your space-time diagram solution?

From Eq. (1) above, we again have $ct' = \gamma ct - \beta\gamma x$, and $x' = -\beta\gamma ct + \gamma x$, where we have identified the S frame as the Earth frame and the S' frame as the rocket's frame. Since $\beta = 0.8$ in this case, then $\gamma = 5/3$ and $\beta\gamma = (4/5)(5/3) = 4/3$. Thus for this particular problem, we have:

$$\begin{aligned} ct' &= \frac{5}{3}ct - \frac{4}{3}x, \text{ and} && (8) \\ x' &= -\frac{4}{3}ct + \frac{5}{3}x && (9) \end{aligned}$$

All that's left to do is to plug the particular coordinates (ct, x) for each event in the Earth frame into Eqs. (8) and (9) to find the corresponding coordinates in the Rocket frame. Let's start with the time coordinates:

$$\begin{aligned} A : (ct, x) = (0, 0) &\implies ct' = \frac{5}{3}(0) - \frac{4}{3}(0) = 0 && (10) \\ B : (ct, x) = (5/6, 2/3) &\implies ct' = \frac{5}{3} \cdot \frac{5}{6} - \frac{4}{3} \cdot \frac{2}{3} = 0.5 && (11) \\ C : (ct, x) = (3/2, 0) &\implies ct' = \frac{5}{3} \cdot \frac{3}{2} - \frac{4}{3}(0) = 2.5 && (12) \\ D : (ct, x) = (15/2, 6) &\implies ct' = \frac{5}{3} \cdot \frac{15}{2} - \frac{4}{3}(6) = 4.5, && (13) \end{aligned}$$

where all the numbers are in units of $c \cdot h$, the distance traveled by light in 1 hour. And now the space coordinates:

$$\begin{aligned} A : (ct, x) = (0, 0) &\implies x' = -\frac{4}{3}(0) + \frac{5}{3}(0) = 0 && (14) \\ B : (ct, x) = (5/6, 2/3) &\implies x' = -\frac{4}{3} \cdot \frac{5}{6} + \frac{5}{3} \cdot \frac{2}{3} = 0 && (15) \\ C : (ct, x) = (3/2, 0) &\implies x' = -\frac{4}{3} \cdot \frac{3}{2} + \frac{5}{3}(0) = -2 && (16) \\ D : (ct, x) = (15/2, 6) &\implies x' = -\frac{4}{3} \cdot \frac{15}{2} + \frac{5}{3}(6) = 0. && (17) \end{aligned}$$

So finally we can write the coordinates of all the events in the Rocket's frame:

$$A : (ct, x) = (0, 0) \implies (ct', x') = (0, 0) \tag{18}$$
$$B : (ct, x) = (5/6, 2/3) \implies (ct', x') = (0.5, 0) \tag{19}$$
$$C : (ct, x) = (3/2, 0) \implies (ct', x') = (2.5, -2) \tag{20}$$
$$D : (ct, x) = (15/2, 6) \implies (ct', x') = (4.5, 0), \tag{21}$$

and we see that all events except C happen at the location of the Rocket ship ($x' = 0$). Below I have copied from Group Problems #6 the space-time diagrams in both the Earth's and Rocket's frame, and we see that indeed the Lorentz-transformed coordinates agree with the space-time diagram, as they must!

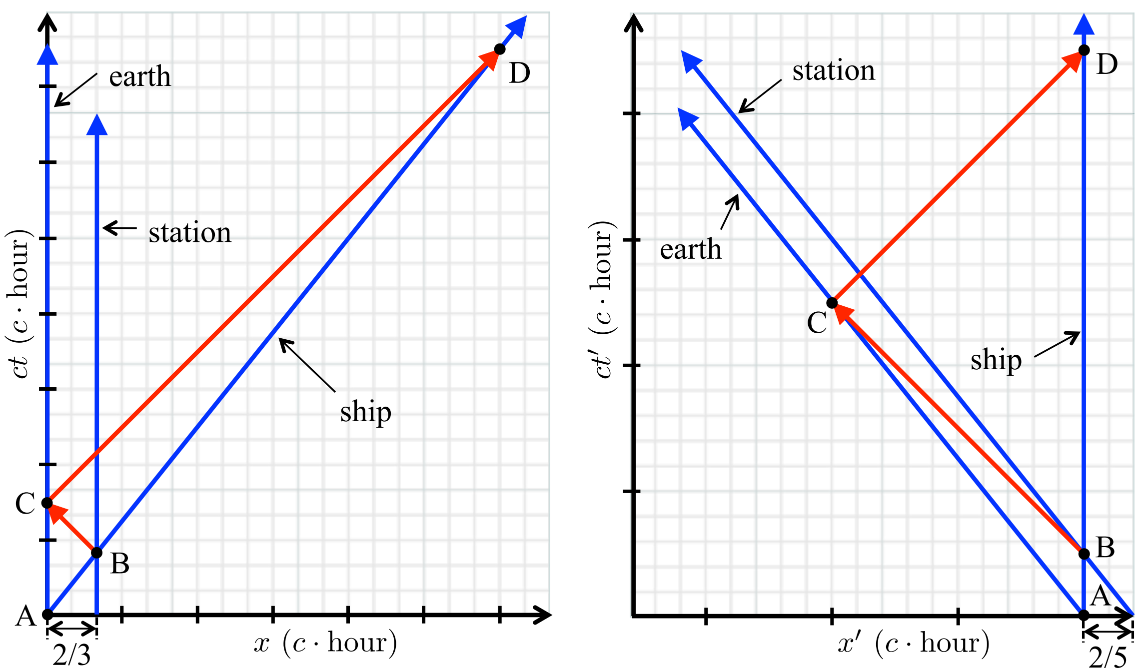
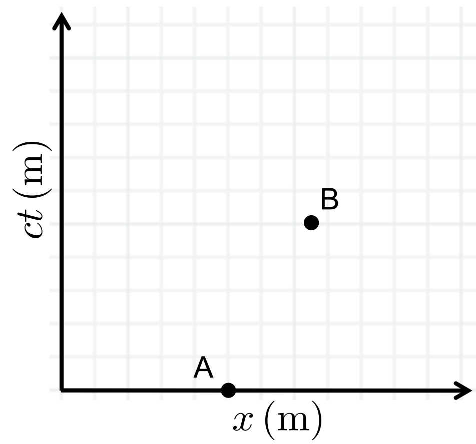


Figure 1: Space-time diagrams in the Earth (left) and rocket ship (right) reference frames.

Problem 3 *Proper time*

Consider two events A$(ct, x) = (0,\ 1\ \text{m})$ and B$(1\ \text{m},\ 3/2\ \text{m})$ in frame $S(ct, x)$.

(a) Plot the events in the $S(ct, x)$ frame.



The time interval between A and B in frame S is $c\Delta t = 1 - 0 = 1$ m and the space interval is $\Delta x = 3/2 - 1 = 1/2$ m. Thus *the* interval is $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 = 1 - 1/4 = 3/4\ \text{m}^2$. Since $(\Delta s)^2 > 0$, A and B are separated by a *time-like* interval and thus we can, in principle, find a frame in which the two events happen at the same location in space; that is, we can find the *proper time interval* between them (see parts b & c).

(b) Find the velocity of the frame $S'(ct', x')$ in which the two events happen at the same place.

There are a number of ways to go about this, including using space-time diagrams. However, the most straightforward is to use the Lorentz transformation for the spatial coordinate: $\Delta x' = -\beta\gamma(c\Delta t) + \gamma(\Delta x)$. (This follows from the second row of the Lorentz Transformation matrix shown in Eq. (1) above.) The problem asks you to find a frame in which the two events happen at the same spatial location, so $\Delta x' = 0$. Thus, we have:

$$\Delta x' = 0 \;=\; -\beta\gamma(c\Delta t) + \gamma(\Delta x) \tag{22}$$

$$= \;-\beta\gamma(1\,\text{m}) + \gamma(0.5\,\text{m}) \tag{23}$$

$$\implies \beta\gamma = 0.5\gamma, \tag{24}$$

and we see that $\beta = 0.5$, or $v = c/2$. So the two events happen at the same spatial location in a reference frame which moves with $v = c/2$ in the $+x$ direction.

(c) What is the proper time separation $\Delta t'$ between A and B?

The proper time is defined as the time interval between two events that happen at the same spatial location in a particular reference frame. There are several ways to go about finding the proper time between events A and B, including using a Lorentz Transformation to find $c\Delta t'$ with $\beta = 0.5$ as we calculated above. A more general and simple way is to use the invariant interval:

$$\begin{aligned} (\Delta s)^2 = (\Delta s')^2 \Longrightarrow (c\Delta t)^2 - (\Delta x)^2 &= (c\Delta t')^2 - (\Delta x')^2 && (25) \\ (1\,\mathrm{m})^2 - (0.5\,\mathrm{m})^2 &= (c\Delta t')^2 - (0)^2 && (26) \\ 3/4\,\mathrm{m}^2 &= (c\Delta t')^2, && (27) \end{aligned}$$

and we see that $c\Delta t' = \sqrt{3}/2$ m, or $\Delta t' = \sqrt{3}/2$ m$/c$. This is the proper time between events A and B, and as such it is the shortest possible time interval between the two events.