Group Problems #6 Solutions

Friday, September 2

Problem 1 *Space-Time Diagram*

At noon a rocket ship passes the Earth at speed $\beta = 0.8$. Observers on the ship and on Earth agree that it is noon. Answer the following questions, and draw complete space-time diagrams in both the Earth and rocket ship frames, showing all events and worldlines:

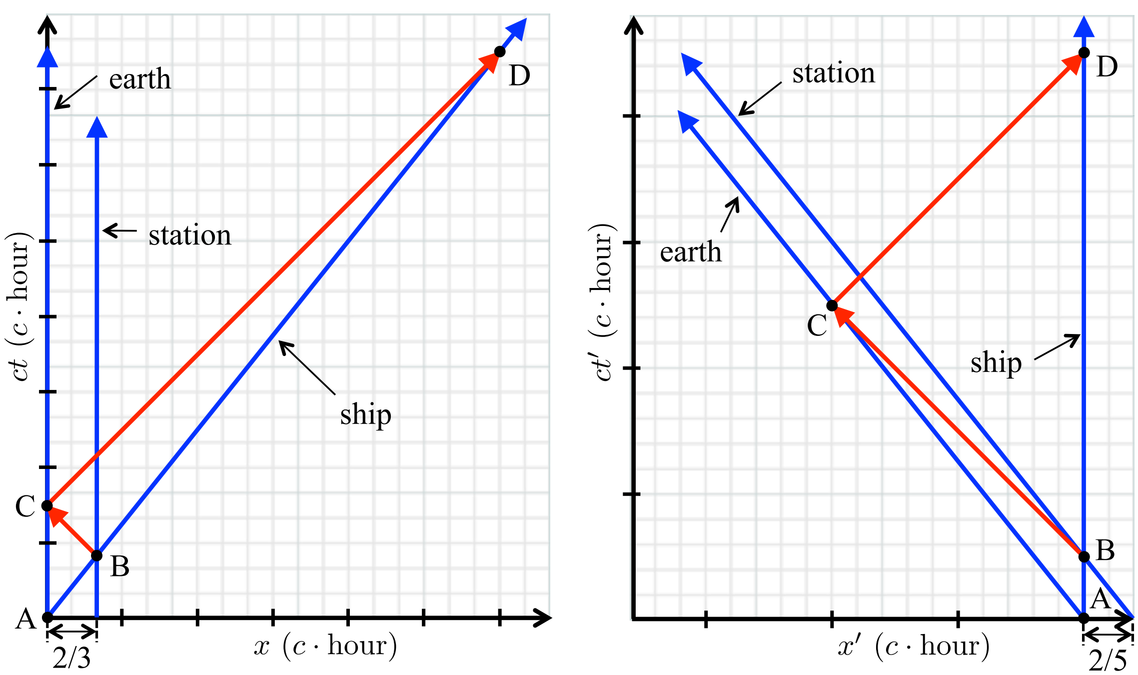

Figure 1: Space-time diagrams in the Earth (left) and rocket ship (right) reference frames.

The space-time diagrams in the Earth and rocket ship frames are shown above. Note that each gridline corresponds to $(c \cdot h)/3$ in the Earth frame and $(c \cdot h)/5$ in the rocket ship frame. One $c \cdot h$ is a unit of distance equal to the distance traveled by light in 1 hour ($3 \cdot 10^5$ km/s $\times 3600$ s $= 1.08 \cdot 10^9$ km.) The four relevant events have been marked: A (ship passes earth - this marks noon); B (ship passes station and sends signal to earth); C (earth relays signal towards ship); and D (ship receives relay). Also note that $\beta = 4/5$ yields $\gamma = 5/3$.

(a) At 12:30 p.m., as read by a rocket ship clock, the ship passes an interplanetary navigational station that is fixed relative to the Earth and whose clocks read Earth time. What time is it at the station? The two events A and B happen at the same location in the ship's frame, so the time between them ($1/2\,h$) is the proper time interval. We can find the interval in the Earth frame via the time dilation formula: $\tau = \gamma\tau_0 = (5/3)(1/2) = 5/6$ hour or 50 minutes. Thus, when the ship passes the station, the station's clock reads 12:50.

(b) How far from Earth, in Earth coordinates, is the station? In the Earth frame, it takes 5/6 of an hour for the ship, moving at speed $v = 4c/5$, to get to the station. Thus, the distance to the station is $d = (4c/5)(5/6\,h) = 2/3\ c \cdot h$.

(c) At 12:30 p.m. rocket time, the ship reports by radio back to Earth. When does Earth receive this signal (in Earth time)? The signal must travel a distance of $2/3\ c \cdot h$ in the Earth frame. Obviously, this takes the light 2/3 of an hour (since it travels at speed c), or 40 minutes. So the signal is received on earth at 12:50 + 40 minutes or 13:30.

(d) The earth replies immediately. When does the rocket receive the response (in rocket time)? In the 1.5 hours it takes for the earth to receive the signal, the rocket travels a distance $(3/2 \text{ hour})(4c/5) = 6/5\ c \cdot h$ (in the Earth frame). This is the separation between the Earth and rocket when the signal is relayed. The relative velocity between the signal and the ship is $c - v = c - 4c/5 = c/5$, so the time it takes (in the Earth frame) for the light signal to catch up with the ship is $\Delta t = (6/5\ c \cdot h)/(c/5) = 6$ hours. If we add this to 13:30 (the time the Earth relays the signal), we see that the ship receives the signal at 19:30 Earth time.

The two events A and D happen at the same spatial location in the rocket ship's frame (at the ship itself) and thus the time interval between them is the proper interval. Thus, we can use the time-dilation formula to convert the time interval in the Earth frame to the proper time interval in the ship's frame. In the Earth frame, the time interval between events A and D is $\tau = 7.5$ hours, so in the ship's frame, it is $\tau_0 = \tau/\gamma = 7.5/(5/3) = 4.5$ hours. Adding this to noon, we see that the light signal is received at 16:30 in ship time.

Problem 2 *Boosting Frames*

Two flashlamps are placed a distance $2L_0$ apart along the $x-$axis in the laboratory frame. At a particular moment in time, the lamps flash simultaneously (events A and B) sending out light pulses, which cross at the midway point (event C), and then arrive simultaneously at the opposite side (events D and E).

(a) Draw a space-time diagram of the situation in the laboratory reference frame. Draw the worldlines of the lamps and the light pulses. Mark all five events and draw lines of simultaneity connecting events A and B and events D and E.

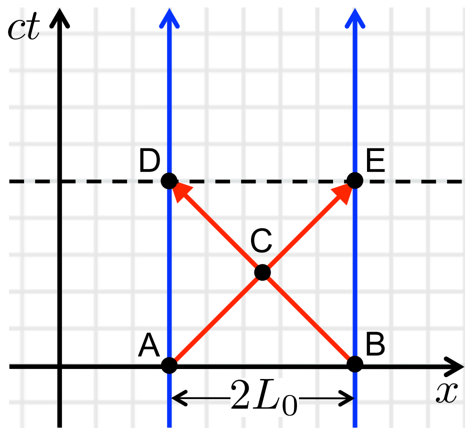

Figure 2: Space-time diagram in the laboratory frame.

(b) Draw a space-time diagram of the situation in a "boosted" frame moving with velocity $v = +\beta c$ relative to the laboratory. Draw all worldlines, events, and the lines of simultaneity that connect events A and B and events D and E.

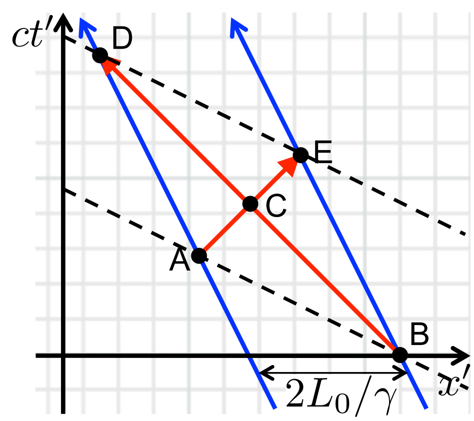

Figure 3: Space-time diagram in the boosted frame.

(c) In the boosted frame, what is the slope of the worldlines corresponding to the lamps (lines of constant position in the lab frame)? What is the slope of the lines connecting events A and B and events E and E (lines of simultaneity in the lab frame)? The lines of constant position (worldlines) obtain a slope of $-1/\beta$. The lines of simultaneity obtain a slope of β. In this example, I chose $\beta = 0.5$, so the worldlines have a slope of -2 and the lines of simultaneity, a slope of $-1/2$.

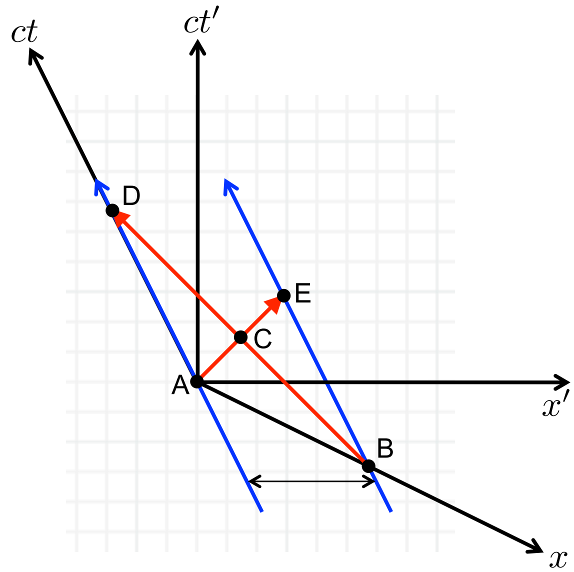

Figure 4: Space-time diagram showing both the laboratory (unprimed) and the boosted frame from the point-of-view of the boosted frame.

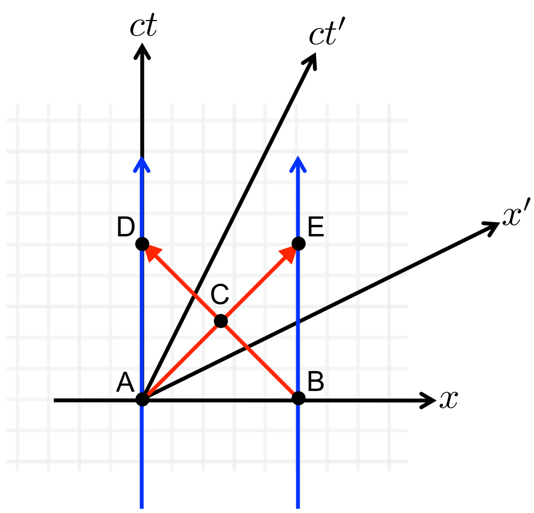

Figure 5: Space-time diagram showing both the laboratory (unprimed) and the boosted frame from the point-of-view of the laboratory frame.

(d) Use this information to draw both reference frames on a single space-time diagram. (See Hogg and/or see me!)

The boost transformation is equivalent to shearing space-time such that lines of constant position adopt a slope of $\pm 1/\beta$ and lines of simultaneity adopt a slope of $\pm\beta$. In Fig. 5, the lines of simultaneity in the primed frame are all parallel to the x' axis, so we see directly in that frame that event B happens before event A (B is below the x'-axis), while E happens before D.