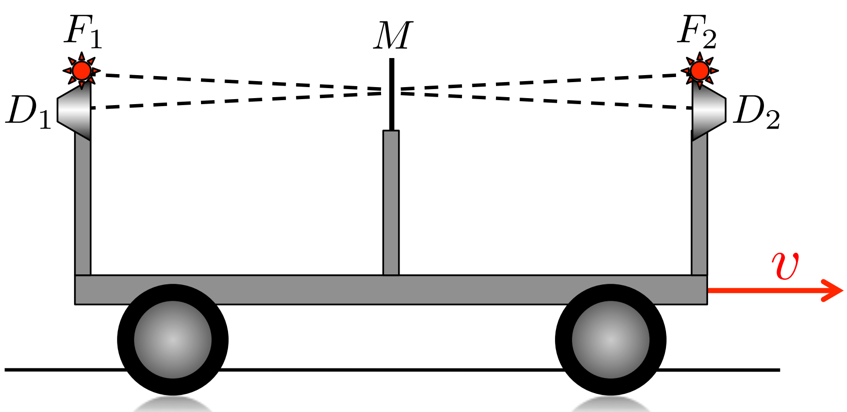
Group Problems #5 Solutions

Wednesday, August 31

Problem 1 *Time dilation and Length Contraction*

A railcar travels to the right with speed v, as shown in the Figure. The proper length of the railcar is $L_0 = 6$ m. At either end of the car, there are flashlamps F_1 and F_2, and at the exact center of the car there is a double-sided mirror M, which reflects the pulses, where they are detected by D_1 and D_2.



(a) How fast must the car go so that its length in the laboratory frame decreases by 1% compared to the proper length? Leave your answer in terms of c. Since the car is moving through the laboratory frame, its length will be contracted by the Lorentz factor, γ:

$$L_{\text{lab}} = \frac{L_0}{\gamma}, \tag{1}$$

$$\text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}} \text{ and } \beta = \frac{v}{c}. \tag{2}$$

Since $0 \leq \beta \leq 1$, then $\gamma \geq 1$. If we write the question in the form of an equation, we have:

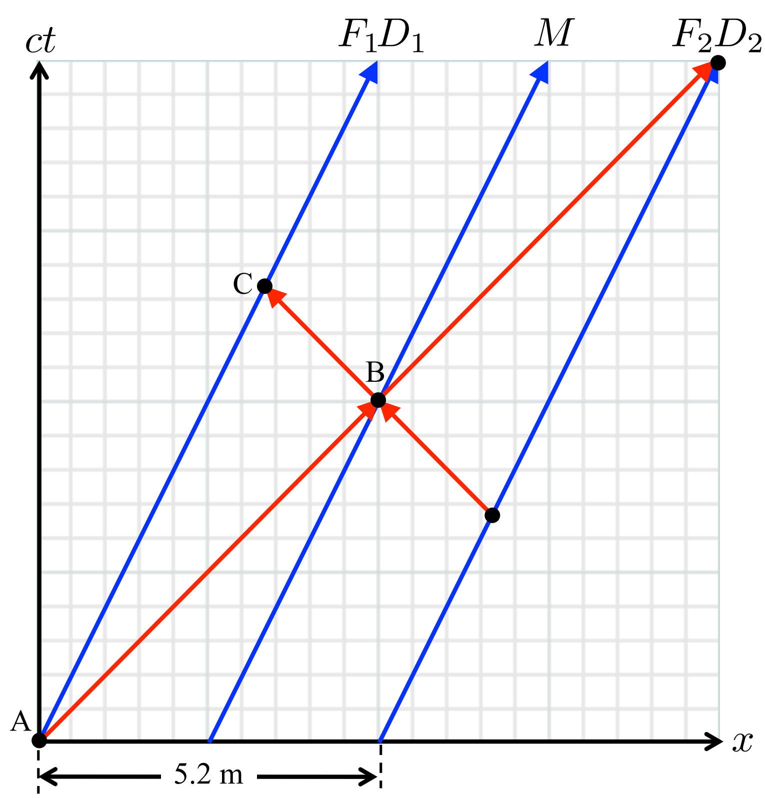
$$\frac{L_0 - L_{\text{lab}}}{L_0} = 0.01 \Longrightarrow \frac{L_{\text{lab}}}{L_0} = 0.99 = \frac{1}{\gamma}. \tag{3}$$

Thus we find that $\sqrt{1-\beta^2} = 0.99 \Longrightarrow \beta = 0.14$. The car must travel at 14% the speed of light (very fast!) to experience a length contraction of only 1% in the laboratory frame!

(b) If $v = c/2$, how much time does it take in the lab frame for a light pulse to travel from F_1 to M and back to D_1. In the car's frame, the light pulse travels a distance of 6 m for the roundtrip, so it takes a time $\Delta t' = (6\,\text{m})/(3\times 10^8$ m/s$)$ $= 2\times 10^{-8}$ s to cover the distance. Since the two events (pulse emission, pulse detection) occur at the same spatial location in the car's frame, this time interval constitutes the *proper time* between the two events. Note that I have implicitly set the car's frame to the S' frame, which is consistent with our usual convention that the primed frame moves to the right with velocity v.

Now we must apply the time dilation formula, $\Delta t = \gamma \Delta t'$, where now the unprimed interval Δt is the time between the two events as measured in the lab frame. The question stipulates that $\beta = 0.5$, so we calculate $\gamma = 2/\sqrt{3}$. Thus, in the lab frame, it takes $(2/\sqrt{3})(2\times 10^{-8})$ s $= 2.31\times 10^{-8}$ s, which is about 15% longer than in the car's frame.

(c) Draw a *neat and precise* space-time diagram of the situation in part (b) as observed in the lab frame. Draw the trajectories of both light pulses, which are emitted simultaneously in the car's frame. **Instead of t on the vertical axis, plot $c \cdot t$.** What is the slope of a light line on such a diagram?



The slope of a light line on a space-time diagram (ct vs. x) is ct/x. Since the distance traveled by a light pulse in time t is $x = ct$, then the slope is clearly unity. Thus, all other worldlines (trajectories) on the space-time diagram must have a (dimensionless) slope that is greater than 1, and in fact the slope of any worldline on a ct vs. x diagram is equal to $1/\beta$.

In this case, we have $\beta = 0.5$, so the slope of worldlines corresponding to the front, middle, and back of the cart must be equal to 2. Note that the length of the cart in the laboratory frame is contracted by the Lorentz factor: $\gamma = 2/\sqrt{3} = 1.15$. So $L_{\text{lab}} = 6/1.15 = 5.2$ m. Thus, the horizontal distance between worldlines corresponding to the front and back of the cart must be 5.2 m.

(d) In the lab frame, does it take more, less, or an equal amount of time for a pulse to go from F_1 to M compared to the time it takes for the pulse to return from M to D_1? Use the space-time diagram in part (c) to discuss your answer. The space-time diagram shows explicitly that the time for the pulse to travel from F_1 to M is longer than the travel time from M to D_1. This is manifest by comparing the vertical coordinates of the three events corresponding to A: emission of the pulse at F_1; B: reflection at M; and C: detection at D_1. The scale at the bottom defines the distance for each gridline on the graph: 5.2 m per 10 gridlines $\Longrightarrow$ 0.52 m/gridline, so we can read the time coordinates for A, B, and C off the graph. In particular, event A occurs at $ct_A = 0$. Event B occurs at 10 gridlines, so $ct_B = 5.2$ m. Finally, $ct_C = 13.4 * 0.52 = 6.97$ m. So,

$$c(t_B - t_A) = 5.2 \text{ m} \quad \Longrightarrow \quad t_B - t_A = 1.73 * 10^{-8} \text{ s}, \tag{4}$$

$$c(t_C - t_B) = 6.97 - 5.2 = 1.77 \text{ m} \quad \Longrightarrow \quad t_C - t_B = 5.9 * 10^{-9} \text{ s}. \tag{5}$$

Also note that $c(t_C - t_A) = 6.97$ m $\Longrightarrow t_C - t_B = 2.32 * 10^{-8}$ s, in agreement with the time dilation calculation in part (b) - within rounding error.