Group Problems #33 - Solutions

Friday, November 17

Problem 1 *Spherical Coordinates*

Show that:

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}.$$

The easiest way to proceed is to use the identities:

$$\hat{r} = \frac{\partial\vec{r}/\partial r}{|\partial\vec{r}/\partial r|}$$

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|}$$

$$\hat{\phi} = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|},$$

where $\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$. Computing the derivatives for $\hat{r}$ gives:

$$\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\left|\frac{\partial\vec{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = 1$$

$$\Longrightarrow \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}.$$

Similarly, for $\hat{\theta}$:

$$\begin{aligned}
\frac{\partial \vec{r}}{\partial \theta} &= r\cos\theta\cos\phi\,\hat{x} + r\cos\theta\sin\phi\,\hat{y} + r\sin\theta\,\hat{z} \\
\left|\frac{\partial \vec{r}}{\partial \theta}\right| &= \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta} = r \\
\Longrightarrow \hat{\theta} &= \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} + \sin\theta\,\hat{z}.
\end{aligned}$$

And finally, for $\hat{\phi}$:

$$\begin{aligned}
\frac{\partial \vec{r}}{\partial \phi} &= -r\sin\theta\sin\phi\,\hat{x} + r\sin\theta\cos\phi\,\hat{y} \\
\left|\frac{\partial \vec{r}}{\partial \phi}\right| &= \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi} = r\sin\theta \\
\Longrightarrow \hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}.
\end{aligned}$$

Problem 2 *Quantization of angular momentum for a satellite*

You and your friend won a NASA grant to do some quantum mechanics experiments on the international space station. After a long day, you decide to go for a space walk. The space station has an orbital radius of ~7,000 km and a speed of ~7.5 km/s. How many allowed values for the z-component of your angular momentum are there? You can use your actual mass, or can assume you have ballooned to a hefty mass of 100 kg due to your rigorous exercise regimen, which consists entirely of space beer-pong (or is it beer space-pong?).

The total angular momentum for a $m = 100$-kg object orbiting at a radius of $r = 7,000$ km at a speed of $v = 7.5$ km/s is given by $mvr = 100 * 7.5 * 7,000 = 5.25 \times 10^{12}$ kg-m^2/s $= 5.25 \times 10^{12}$ J-s. According to the postulates of quantum mechanics, the total angular momentum is equivalent to:

$$\begin{aligned}
\left|\vec{L}\right| &= \sqrt{\ell(\ell+1)}\hbar \simeq \ell\hbar \text{ (for } \ell \gg 1) \\
\Longrightarrow \ell &\simeq \frac{L}{\hbar} = \frac{5.25 \times 10^{12}}{1.05 \times 10^{-34}} = 5 \times 10^{46}.
\end{aligned}$$

So ℓ is a very large integer! There are $2\ell+1 = 10^{47}$ possible values for the z-component of the angular momentum.