Group Problems #32 - Solutions

Wednesday, November 15

Problem 1 *Normalization of the 3D Well Wavefunctions*

Find the normalization constant for the 3D infinite square well. You may want to use the trigonometric identity: $\cos 2\theta = 1 - 2\sin^2\theta$.

The pre-normalized wave function for the 3D infinite well is just the product of 1D solutions for the different directions:

$$\psi_{n_x,n_y,n_z}(x,y,z) = A \sin\frac{n_x \pi x}{L_x} \sin\frac{n_y \pi y}{L_y} \sin\frac{n_z \pi z}{L_z}. \tag{1}$$

Since the x, y and z directions are independent, we can also write $A = A_x A_y A_z$, where $A_x = \sqrt{2/L_x}$, $A_y = \sqrt{2/L_y}$, and $A_z = \sqrt{2/L_z}$. Thus, we have:

$$A = \sqrt{\frac{8}{L_x L_y L_z}}. \tag{2}$$

(a) How does it depend on the quantum number, n_x, n_y, and n_z?

Clearly it doesn't depend on n_x, n_y and n_z. This is because the size of the well is constant and independent of energy. This is not the case for the harmonic oscillator potential, for example, for which there is a different normalization constant for each energy state.

(b) How does it depend on the length of each side, L_x, L_y, L_z?

We see that A^2 has dimensions of $length^{-3} \equiv 1/volume$. This makes sense since the normalization integral for $\psi_{n_x,n_y,n_z}(x,y,z)$ is over all three dimensional space, $\int \psi^* \psi \, dV$, and this integral must be unitless because it represents a probability.

Problem 2 *Space-Pong*

You and your friend have won a NASA grant to do some quantum mechanics experiments on the international space station. During a break, you play a game of space-pong, which is similar to ping-pong except that it is played on a table as long as it is wide and in a room with a ceiling that is this same distance above the table (there's not a lot of room on the space station). You and your friend are excellent players and neither of you ever lets the ball get past you.

(a) What is the ground-state energy for the ping-pong ball if it has a mass of 2×10^{-3} kg, and the length of the table is 2 m?

The ground state of the 3D infinite well is:

$$\begin{aligned} E_{111} &= (1^2 + 1^2 + 1^2)\frac{\pi^2\hbar^2}{2mL^2} && (3) \\ & && (4) \\ &= 3\frac{\pi^2(1.05 \times 10^{-34}\ \text{kg}\,\text{m}^2/\text{s})^2}{2(2 \times 10^{-3}\ \text{kg})(4\ \text{m}^2)} && (5) \\ & && (6) \\ &= 2.04 \times 10^{-65}\ \text{J} && (7) \\ & && (8) \\ &= 1.3 \times 10^{-46}\ \text{eV}, && (9) \end{aligned}$$

a very small energy indeed!

(b) Estimate the probability that the ball will be found in the ground state during a game of space-pong. Show your work/reasoning.

Any reasonable velocity for the ball will yield a kinetic energy that is many, many orders of magnitude larger than E_{111}, so the ping-pong ball is in a very high energy state, $n_x, n_y, n_z \gg 1$. For example, imagine that it takes the ping-pong ball 1/5 second to cross the 2 m table, corresponding to a velocity of 10 m/s, and a kinetic energy of $10^{-3} \cdot 10^2 = 0.1$ J. Thus, the probability of the particle being found in the ground state is vanishingly small.

(c) Would you expect the ground-state energy for the ball to go up or down if the game were played on Earth instead of on the space station? Give your reasoning.

Since the gravitational potential energy takes the form, $U(z) = mgz$, then on earth, gravity provides additional confinement for the ping-pong ball in the z direction. *Any* time the potential is more confined, the energy levels, including the ground state, must increase. This follows from the uncertainty principle, and would also follow explicitly if you were to solve the Shrodinger equation for this potential.