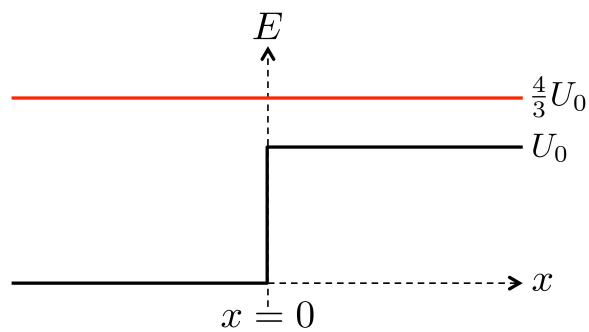
Group Problems #29 - Solutions

Monday, November 7

Problem 1 *Step potential with $E > U_0$*

A particle of energy E is incident from the left onto a finite potential step of height U_0. Consider the case when $E = \frac{4}{3}U_0$.

(a) Draw the potential energy diagram, including the energy of the particle. Do your best to draw this to scale.



(b) What is the ratio of the wavelengths of the wavefunction of the transmitted (t) and incident (i) waves, λ_t/λ_i?

When a particle moves into the region $x > 0$, its kinetic energy must decrease (and thus its momentum and velocity must also decrease) since its potential energy increases and since total energy is conserved. A particle's momentum is characterized by its wave vector: $p = \hbar k = h/\lambda$. Thus we expect the wavelength to *increase* in the region $x > 0$. The kinetic energy is given by:

$$K = E - U(x) = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \Longrightarrow k = \frac{\sqrt{2m[E - U(x)]}}{\hbar}, \tag{1}$$

where $U(x) = 0$ when $x < 0$, $U(x) = U_0$ when $x > 0$, and $E = \frac{4}{3}U_0$. Taking this all into consideration, the incident wave vector in the region $x < 0$ is given by:

$$k_i = \frac{2\pi}{\lambda_i} = \frac{\sqrt{2mE}}{\hbar}, \tag{2}$$

and the transmitted wave vector in the region $x > 0$ is given by:

$$k_t = \frac{2\pi}{\lambda_t} = \frac{\sqrt{2m(E-U_0)}}{\hbar}. \tag{3}$$

Using these expressions, we can calculate the ratio of wavelengths for the incident and transmitted beam of particles:

$$\frac{k_i}{k_t} = \frac{2\pi/\lambda_i}{2\pi/\lambda_t} = \frac{\lambda_t}{\lambda_i} \quad = \quad \frac{\sqrt{2mE}/\hbar}{\sqrt{2m(E-U_0)/\hbar}} \tag{4}$$

$$\tag{5}$$

$$= \quad \sqrt{\frac{E}{E-U_0}} = \sqrt{\frac{4/3\,U_0}{1/3\,U_0}} = 2. \tag{6}$$

(c) What is the ratio of the amplitudes of the wavefunction of the transmitted and incident waves, A_t/A_i?

The spatial wave function to the left (L) of the barrier ($x < 0$) is the superposition of right-moving and left-moving plane waves corresponding to the incident and reflected particle beams, respectively:

$$\psi_L(x) = A_i e^{ik_i x} + A_r e^{-ik_r x}, \tag{7}$$

where $k_i = k_r = \sqrt{2mE}/\hbar$. Similarly, the spatial wave function to the right (R) of the barrier ($x > 0$) is just a right moving plane wave corresponding to the transmitted beam:

$$\psi_R(x) = A_t e^{ik_t x}, \tag{8}$$

where $k_t = \sqrt{2m(E-U_0)}\hbar$. Applying the appropriate boundary conditions, namely continuity of the wave function and its 1st spatial derivative at $x = 0$ leads to the expression (see Lecture 29):

$$\frac{A_t}{A_i} = \frac{2k_i}{k_i + k_r} \quad = \quad \frac{2\sqrt{E}}{\sqrt{E} + \sqrt{E-U_0}} \tag{9}$$

$$\tag{10}$$

$$= \quad \frac{2\sqrt{\frac{4}{3}U_0}}{\sqrt{\frac{4}{3}U_0} + \sqrt{\frac{1}{3}U_0}} \tag{11}$$

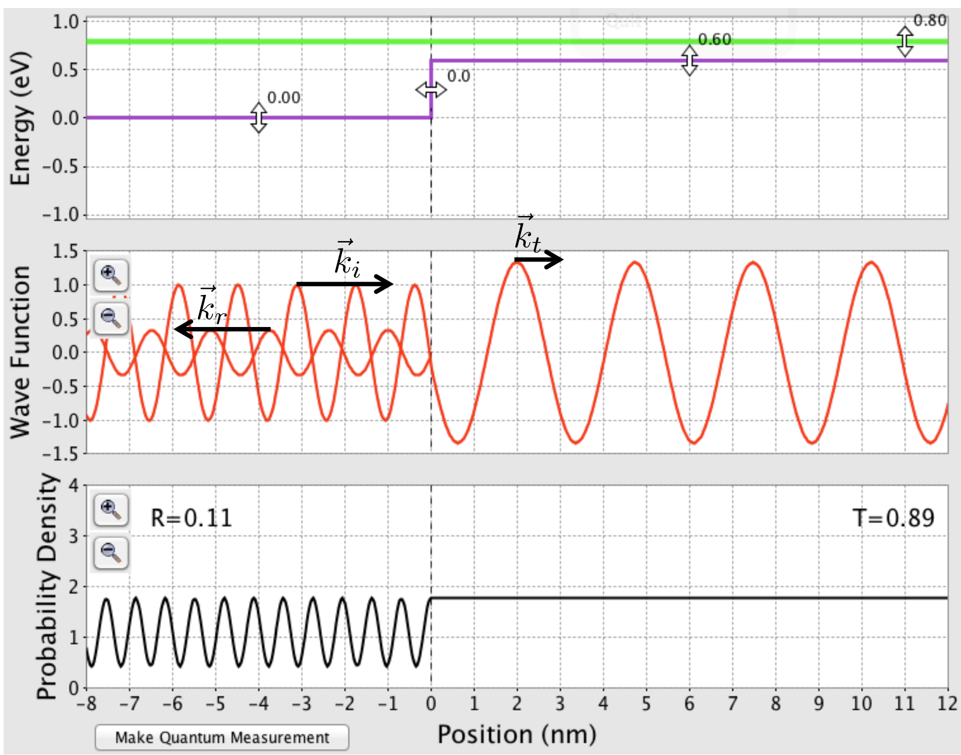
$$\tag{12}$$

$$= \quad \frac{4\sqrt{\frac{1}{3}}}{2\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}}} = \frac{4}{3}. \tag{13}$$

Does this make sense? On the one hand, you might expect that $A_t/A_i < 1$ since only a fraction of the incident particles are transmitted to the right of the barrier (see part (d) below). However, each particle that is transmitted slows down, and thus the particles within the beam "bunch up" to the right of the barrier. The important thing to remember is that the amplitude coefficients are related to the density of particles within the beam, $A^2 = \rho(x)$, where $\rho(x)$ is the number of particles per unit length at position x. Thus, as the particles bunch up to the right of the barrier, they get closer together and $\rho(x)$ increases for $x > 0$.

(d) Draw the incident and transmitted wavefunctions superimposed on the graph from part (a). Again, do your best to draw this to scale.

Below is a screen capture from the Java simulation posted on our website. The incident, reflected, and transmitted waves have been indicated by $\vec{k}_i$, $\vec{k}_r$, and $\vec{k}_t$.



(e) What are the transmission and reflection coefficients, $\mathcal{T}$ and $\mathcal{R}$?

The reflection coefficient is given by:

$$\mathcal{R} = \frac{|A_r|^2}{|A_i|^2} = \left(\frac{k_i - k_t}{k_i + k_t}\right)^2 = \left(\frac{\sqrt{4/3} - \sqrt{1/3}}{\sqrt{4/3} + \sqrt{1/3}}\right)^2 = \left(\frac{2-1}{2+1}\right)^2 \tag{14}$$

$$\tag{15}$$

$$= \frac{1}{9} \simeq 0.11, \tag{16}$$

and the transmission coefficient is then given by $\mathcal{T} = 1 - \mathcal{R} = \frac{8}{9} \simeq 0.89$.