Group Problems #28 - Solutions

Friday, November 4

Problem 1 *Uncertainty principle and the Harmonic Oscillator*

Consider the ground state of the harmonic oscillator potential with $\psi_0(x) = Ae^{-ax^2}$, where $A = (m\omega_0/\pi\hbar)^{1/4}$, and $a = m\omega_0/2\hbar$.

Find the ground-state energy, E_0, using only the Heisenberg uncertainty principle and the general expressions for the uncertainties in x and p,

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 \tag{1}$$
$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2. \tag{2}$$

(a) Show that the expectation value for both the position and momentum of the ground state are both zero, $\langle x\rangle = \langle p\rangle = 0$. (You can show this explicitly or use appropriate symmetry arguments.)

The expectation value for position is given by:

$$\langle x\rangle = \int_{-\infty}^{\infty} Ae^{-ax^2}\, x\, Ae^{-ax^2}\, dx = A^2 \int_{-\infty}^{\infty} x\, e^{-2ax^2}\, dx = 0, \tag{3}$$

where we have used the fact that an integral of an odd-symmetry function over symmetric bounds is zero. Similarly, the expectation value for momentum is given by:

$$\langle p\rangle = \int_{-\infty}^{\infty} Ae^{-ax^2}\, (-i\hbar)\frac{\partial}{\partial x}\, Ae^{-ax^2}\, dx = 2ia\hbar A^2 \int_{-\infty}^{\infty} x\, e^{-2ax^2}\, dx = 0. \tag{4}$$

(b) Use the result from part (a) to find an expression for the expectation value of the energy, $\langle E\rangle = E_0$, in terms of Δx and Δp.

The expectation value of the energy is obtained by applying the Hamiltonian operator $\hat{H}$ to the wave function:

$$\langle E\rangle = \left\langle \psi \left| \hat{H} \right| \psi \right\rangle = \int_{-\infty}^{\infty} \psi^*(x) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_0^2 x^2 \right) \psi(x)\, dx \tag{5}$$

$$\tag{6}$$

$$= \int_{-\infty}^{\infty} \psi^*(x) \left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\, \hat{x}^2 \right) \psi(x)\, dx \tag{7}$$

$$\tag{8}$$

$$= \frac{\langle p^2\rangle}{2m} + \frac{1}{2}m\omega_0^2 \left\langle x^2 \right\rangle, \tag{9}$$

where we have used $\hat{p} = -i\hbar\partial/\partial x \longrightarrow \hat{p}^2 = -\hbar^2\partial^2/\partial x^2$, and $\hat{x}^2 = x^2$. Also, the so-called "bra-ket" (bracket) notation has been introduced in Eqn. 5. In this case, we are interested in the expectation value for the lowest energy (ground) state, so we can use the the results from part (a) and Eqs. 1 & 2 to substitute for $\langle p^2\rangle$ and $\langle x^2\rangle$:

$$\langle E\rangle = \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega_0^2 (\Delta x)^2 \geq E_0, \tag{10}$$

where we have indicated that the expression in Eqn. 10 is greater than or equal to the true ground state energy E_0 - it will only be equal to E_0 for some optimized values of Δp and Δx.

(c) Use the result from part (b) and the *exact form* of the uncertainty relation to show that,

$$E_0 \geq \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega_0^2(\Delta x)^2 \tag{11}$$

The idea here is to write Eqn. 10 in terms of either Δp or Δx only. So we write:

$$\Delta x\, \Delta p \geq \frac{\hbar}{2} \tag{12}$$

$$\tag{13}$$

$$\Longrightarrow \Delta p = \frac{\hbar}{2\Delta x} \tag{14}$$

$$\tag{15}$$

$$\Longrightarrow (\Delta p)^2 = \frac{\hbar^2}{4(\Delta x)^2}, \tag{16}$$

and we have adopted the equality (which is justified since the ground state wave function is Gaussian). Making the substitution for $(\Delta p)^2$ in Eqn. 10 gives:

$$\langle E\rangle = \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega_0^2(\Delta x)^2 \geq E_0. \tag{17}$$

Note that the wording of the problem may have been a little confusing in that E_0 is the true ground state energy, while $\langle E\rangle$ is the expectation value of the energy, which must be greater than or equal to E_0.

(d) Find the value of $(\Delta x)^2$ corresponding to the minimum value of the right hand side of Eq. (11), and then insert this back into Eq. (11). This gives the smallest value of E_0 consistent with the uncertainty principle, which is the answer.

Here we just take the derivative of $\langle E \rangle$ with respect to Δx and set that equal to zero: this will find the value of Δx corresponding to the minimum value of $\langle E \rangle$. Taking the derivative gives:

$$\frac{d\langle E \rangle}{d(\Delta x)} = -\frac{\hbar^2}{4m(\Delta x)^3} + \frac{1}{2}m\omega_0^2(2\Delta x) = 0 \tag{18}$$

$$\tag{19}$$

$$\Longrightarrow \quad m\omega_0^2 \Delta x = \frac{\hbar^2}{4m(\Delta x)^3} \tag{20}$$

$$\tag{21}$$

$$\Longrightarrow \quad (\Delta x)^4 = \frac{\hbar^2}{4m^2\omega_0^2} \Longrightarrow (\Delta x)^2 = \frac{\hbar}{2m\omega_0}. \tag{22}$$

Substituting for $(\Delta x)^2$ in Eqn. 17 gives:

$$\langle E \rangle = \frac{\hbar^2}{8m}\frac{\hbar}{2m\omega_0} + \frac{1}{2}m\omega_0^2\frac{\hbar}{2m\omega_0} \tag{23}$$

$$\tag{24}$$

$$= \frac{1}{4}\hbar\omega_0 + \frac{1}{4}\hbar\omega_0 \tag{25}$$

$$\tag{26}$$

$$= \frac{1}{2}\hbar\omega_0 = E_0. \tag{27}$$

With this problem, we have shown that the uncertainly principle can be used to determine the exact ground state energy of the harmonic oscillator potential. This is possible since the ground state wave function is Gaussian, which is the limiting (exact) case for the uncertainty principle.