Group Problems #27 - Solutions

Wednesday, November 2

Problem 1 *Gaussian solution to the harmonic oscillator*

As we'll see, the ground-state of the quantum harmonic oscillator has a Gaussian wavefunction of the form $\psi(x) = Ae^{-ax^2}$.

(a) What are the units (dimensions) of the constant a?

The argument of the exponential function must be dimensionless (otherwise, what would be the dimension of the exponential function itself?). Thus, a must have units of 1/length^2: $a \equiv L^{-2}$.

(b) Use the kinetic energy operator $\hat{K}$ to find an expression for the constant a in terms of the classical turning point, x_T, where a classical oscillator (e.g., mass on a spring) would change direction.

The kinetic energy operator is defined as:

$$\hat{K} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}, \tag{1}$$

where $\hat{K}$ acting on the wave function gives the *value* of the kinetic energy $K(x)$ at a particular position x multiplied by the wave function: $\hat{K}\psi(x) = K(x)\psi(x)$. Applying this to the wave function above gives:

$$\hat{K}\psi(x) = K(x)\psi(x) \implies -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[Ae^{-ax^2}\right] = \frac{\hbar^2 a}{m}Ae^{-ax^2}[1-2ax^2] \tag{2}$$

$$\implies K(x) = \frac{\hbar^2 a}{m}[1-2ax^2]. \tag{3}$$

At the classical turning point, $x = x_T$, the kinetic energy is zero, so

$$K(x = x_T) = 0 \implies 1 - 2ax_T^2 = 0 \tag{4}$$

$$\implies a = \frac{1}{2x_T^2}, \tag{5}$$

and we see that a has units of 1/length^2 as required. Note that $E = U(x)$ at $x = x_T$, and there is an inflection point in $\psi(x)$.

(c) Consider a particle in the ground state of the harmonic oscillator potential. If you were to make a measurement of its kinetic energy K at a particular point in space, then you must get a number. If you repeat this measurement over and over again, will you get the same number? Why or why not? (*Hint*: think about the commutation relationship between the position operator $\hat{x}$ and the kinetic energy operator $\hat{K}$.)

Using the hint, we should calculate the commutator of $\hat{K}$ and $\hat{x}$: $[\hat{K}, \hat{x}] = \hat{K}\hat{x} - \hat{x}\hat{K}$, where:

$$\hat{K} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \tag{6}$$

$$\hat{x} = x. \tag{7}$$

Applying this to the wave function, we have:

$$\hat{K}\hat{x}\,\psi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[x\,Ae^{-ax^2}\right] \tag{8}$$

$$\tag{9}$$

$$= -A\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[e^{-ax^2}(1-2ax^2)\right] \tag{10}$$

$$\tag{11}$$

$$= Ae^{-ax^2}\frac{\hbar^2}{2m}2ax\left(3-2ax^2\right), \tag{12}$$

and

$$\hat{x}\hat{K}\psi(x) = -\frac{\hbar^2}{2m}x\frac{\partial^2}{\partial x^2}\left[Ae^{-ax^2}\right] \tag{13}$$

$$\tag{14}$$

$$= \frac{\hbar^2}{2m}Ax\frac{\partial}{\partial x}\left[2axe^{-ax^2}\right] \tag{15}$$

$$\tag{16}$$

$$= Ae^{-ax^2}\frac{\hbar^2}{2m}2ax\left(1-2ax^2\right). \tag{17}$$

Putting this together gives:

$$[\hat{K}, \hat{x}]\psi(x) = [\hat{K}\hat{x} - \hat{x}\hat{K}]Ae^{-ax^2} \tag{18}$$

$$\tag{19}$$

$$= Ae^{-ax^2}\frac{\hbar^2}{2m}2ax\left[\left(3-2ax^2\right)-\left(1-2ax^2\right)\right] \tag{20}$$

$$\tag{21}$$

$$= Ae^{-ax^2}\frac{\hbar^2}{2m}2ax\cdot 2 \tag{22}$$

$$\tag{23}$$

$$\implies [\hat{K}, \hat{x}] = \frac{2a\hbar^2}{m}x \neq 0. \tag{24}$$

Since this is not equal to zero, the $\hat{K}$ and $\hat{x}$ do not commute, and we cannot simultaneously measure the particle's kinetic energy and position simultaneously. So if we constrain our measurement to a particular value of position (x), then we will measure a spread in kinetic energy values when we repeat the measurement many times on a similarly prepared system.

(d) What is the expectation value for the momentum of a particle in the ground state of the harmonic oscillator? (*Hint*: This question involves doing an apparent integral, but you can use symmetry arguments to avoid actually computing the integral.)

The expectation value of the physical quantity (observable) associated with operator $\hat{O}$ is given by:

$$\langle O \rangle = \int_{-\infty}^{\infty} \psi^*(x)\hat{O}\psi(x)dx. \tag{25}$$

In this case, $\hat{O} = \hat{p} = -i\hbar\frac{\partial}{\partial x}$. So finally,

$$\langle p \rangle = -i\hbar A^2 \int_{-\infty}^{\infty} e^{-ax^2} \frac{d}{dx}\left(e^{-ax^2}\right) dx \tag{26}$$

$$\tag{27}$$

$$= 2i\hbar A^2\, a \int_{-\infty}^{\infty} x e^{-2ax^2} dx = 0, \tag{28}$$

since the argument is an odd function of x integrated over symmetric bounds. Thus, the expectation value of the momentum is zero: this doesn't mean that a measurement of the momentum will equal zero. It means that repeated measurements of the momentum on similarly prepared systems will yield a mean value of zero.