Group Problems #26 - Solutions

Monday, October 31

Problem 1 *Wavefunction curvature in classically forbidden region*

A finite potential energy function $U(x)$ allows $\psi(x)$, the solution of the time-independent Schrödinger equation, to penetrate the classically forbidden region. Without assuming any particular function for $U(x)$, show that $\psi(x)$ must have an inflection point at any value of x where it enters a classically forbidden region.
The time-independent Schrödinger equation is given by:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x) \tag{1}$$

$$\tag{2}$$

$$\Longrightarrow \frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}\left[U(x) - E\right]\psi(x), \tag{3}$$

where E is the total energy of a particle with mass m. The left side of Eqn. 2 represents the *curvature* of the wavefunction $\psi(x)$ (remember your calculus - the first derivative is the slope, the second derivative is the curvature). An inflection point is a point where the curvature changes sign, that is where $d^2\psi(x)/dx^2 = 0$. Inspection of Eqn. 2 shows that this happens when $E = U(x)$ and/or when $\psi(x) = 0$. In general, $\psi(x) \neq 0$, so we must have $E = U(x)$, which is exactly the boundary between the classically allowed ($E > U(x)$) and forbidden ($E < U(x)$) regions.

Problem 2 *Penetration depth, I*

A 50 eV electron is trapped between electrostatic walls 200 eV high. How far does its wavefunction extend beyond the walls?
The penetration depth, δ, is given by:

$$\delta \equiv \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = \frac{\hbar c}{\sqrt{2mc^2(U_0 - E)}}. \tag{4}$$

For the current problem, $E = 50$ eV, and $U_0 = 200$ eV, so $U_0 - E = 150$ eV. Also, $m_e c^2 = 511 \times 10^3$ eV and $\hbar c = 197$ eV·nm. Thus,

$$\delta = \frac{\hbar c}{\sqrt{2mc^2(U_0 - E)}} = \frac{197}{\sqrt{2(511 \times 10^3) \cdot 150}} = 0.016\,\text{nm}. \tag{5}$$

Problem 3 *Penetration depth, II*

An electron is trapped in a finite well. Its wavefunction penetrates into the classically forbidden region by 1 nm. How much energy is required to "free" the electron from the well?

What we want is the difference between the energy of the electron and the top of the potential well, $U_0 - E$. Solving Eqn. 4 for $U_0 - E$, we have

$$U_0 - E = \frac{(\hbar c)^2}{2mc^2\delta^2} = \frac{197^2}{2(511 \times 10^3) \cdot 1^2} = 38\,\text{meV}. \tag{6}$$