Group Problems #25 - Solutions

Friday, October 28

Problem 1 *Superposition state*

A particle is trapped in a 1D infinite square well of length L. It is prepared in a particular state, $\psi(x)$, and a measurement is made of its energy. This process is repeated many times; each time the particle is prepared in the same state, $\psi(x)$. After a large number of measurements, it is found that 50% of the time, the measurement results in a value of $h^2/8mL^2$, 25% of the time in $h^2/2mL^2$, and 25% of the time in $9h^2/8mL^2$.

(a) Write down an explicit expression for the wavefunction $\psi(x)$.

In general, $\psi(x)$ is a superposition of eigenstates:

$$\psi(x) = \sum_{n=1}^{\infty} C_n \psi_n(x). \tag{1}$$

When an energy measurement is made on the wavefunction, a single, real-valued number must be the result, and this number must correspond to one of the eigenenergies, E_n. Furthermore, the probability of obtaining a particular eigenergy, E_n, for a particular measurement is given by the magnitude squared of the normalized coefficient, $|C_n|^2$. The given eigenenergies correspond to $n = 1$, $n = 2$, and $n = 3$. Thus we have $|C_1|^2 = 1/2$, $|C_2|^2 = 1/4$, and $|C_3|^2 = 1/4$. Thus, if we neglect the possibility that the coefficients, C_n, are complex (this is not true in general, but we can stipulate this for a particular problem), we find that $C_1 = 1/\sqrt(2)$, $C_2 = 1/2$, and $C_3 = 1/2$. This gives:

$$\psi(x) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{2}\psi_2(x) + \frac{1}{2}\psi_3(x) \tag{2}$$

$$= \sqrt{\frac{2}{L}}\left[\frac{1}{\sqrt{2}}\sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{3\pi x}{L}\right)\right] \tag{3}$$

(b) What is the expectation value of the energy?

The expectation value of the energy is given by,

$$\langle E \rangle = \sum_{n=0}^{\infty} |C_n|^2 E_n = \frac{h^2}{8mL^2}\left[1^2\left(\frac{1}{\sqrt{2}}\right)^2 + 2^2\left(\frac{1}{2}\right)^2 + 3^2\left(\frac{1}{2}\right)^2\right] \tag{4}$$

$$= \frac{h^2}{8mL^2}\left(\frac{1}{2} + 1 + \frac{9}{4}\right) = \frac{15h^2}{32mL^2}. \tag{5}$$

(c) If a particular measurement of the energy yields a value of $h^2/2mL^2$, what is the probability that the particle can then be found between $x = 0$ and $x = L/4$?

The given eigenenergy corresponds to $n = 2$. Thus, immediately after the measurement, the particle is in the $n = 2$ eigenstate with wavefunction,

$$\psi(x) = \psi_2(x) = \sqrt{\frac{2}{L}} \sin \frac{2\pi x}{L}. \tag{6}$$

Note that although before the measurement, there is only a 25% probability of finding the particle in the $n = 2$ eigenstate, after the measurement there is a 100% probability ($C_1 = C_3 = 0$, and $C_2 = 1$ after the measurement). Thus we only have to integrate $\psi^*\psi$ over $x = 0 \rightarrow L/4$ to get the probability of finding the particle in that region. We can also do this by inspection of the graph of the probability density $dP/dx = \psi^*\psi$ for $n = 2$ as given in the notes. The answer is 1/4.