Group Problems #24 - Solutions

Wednesday, October 26

Problem 1 *Electron trapped in a 1D well*

An electron is trapped in a 1D region of length 10^{-10} m (a typical atomic diameter).

(a) How much energy must be supplied to excite the electron from the ground state to the first excited state?

The allowed energies are:

$$E_n = \frac{n^2\hbar^2\pi^2}{2mL^2} = \frac{n^2h^2}{8mL^2}, \tag{1}$$

so the ground state energy is:

$$E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240 \text{ eV nm})^2}{8(511\times10^3 \text{ eV})(10^{-2} \text{ nm}^2)} \tag{2}$$

$$\tag{3}$$

$$= 37.6 \text{ eV}. \tag{4}$$

In the first excited state, the energy is $4E_1$. The difference, which must be supplied, is $3E_1 = 113$ eV.

(b) In the ground state, what is the probability of finding the electron in the region from $x = 0.09\times10^{-10}$ m to 0.11×10^{-10} m?

The spatial wavefunctions (eigenfunctions) are given by:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}. \tag{5}$$

For the ground state ($n = 1$), the probability is the integral of the probability density, $dP/dx = \psi_1^*(x)\psi_1(x)$, over the appropriate interval:

$$P = \int_{x_1}^{x_2} |\psi|^2 = \frac{2}{L}\int_{x_1}^{x_2} \sin^2\frac{\pi x}{L}dx \tag{6}$$

$$= \left[\frac{x}{L} - \frac{1}{2\pi}\sin\frac{2\pi x}{L}\right]_{x_1}^{x_2} \tag{7}$$

$$\tag{8}$$

$$= 0.0038 = 0.38\%. \tag{9}$$

To compute the integral in Eq. 6, we have used the double-angle formula: $\cos(2\theta) = 1 - 2\sin^2\theta$.

(c) In the first excited state, what is the probability of finding the electron between $x = 0$ and $x = 0.25 \times 10^{-10}$ m?

The probability is as in part (b) but we now must use the 1^{st} excited-state wavefunction ($n = 2$) and the new interval:

$$\begin{aligned} P &= \frac{2}{L}\int_{x_1}^{x_2} \sin^2\frac{2\pi x}{L} dx && (10) \\ &= \left[\frac{x}{L} - \frac{1}{4\pi}\sin\frac{4\pi x}{L}\right]_{x_1}^{x_2} && (11) \\ & && (12) \\ &= 0.25. && (13) \end{aligned}$$

This result is what we would expect by inspection of the graph of $|\psi|^2$ for $n = 2$, as given in the notes (Lecture 24). The interval from $x = 0$ to $x = L/4$ contains 25% of the total area under the ψ^2 curve.

Problem 2 *Waves on a string solution to particle in a box*

Use DeBroglie's relationship for momentum to show that standing waves on a string of length L (the string is pinned at both ends) have the same energy eigenvalues as a particle in a 1D box (infinite well).

For a standing wave on a string pinned at both ends (e.g., a guitar string), the lowest energy ($n = 1$) state has one antinode in the middle. In other words, $\lambda_1/2 = L \Rightarrow \lambda_1 = 2L$, where λ is the wavelength of the fundamental vibrational mode and L is the distance between the two ends of the string. The first overtone ($n = 2$) has two antinodes, so $\lambda_2 = L \Rightarrow \lambda_2 = L$, and the second overtone ($n = 3$) has three antinodes, so $3\lambda_3/2 = L \Rightarrow \lambda_3 = 2L/3$. So the pattern is captured by the formula:

$$\lambda_n = \frac{2L}{n}, \tag{14}$$

where n is an integer 1 or greater. Now we use DeBroglie's relationship, $p_n = h/\lambda_n$, and the classical expression for the total mechanical energy (when the potential energy is zero):

$$E = \frac{p^2}{2m} \Longrightarrow E_n = \frac{p_n^2}{2m} = \frac{n^2h^2}{4L^2}\frac{1}{2m} = \frac{n^2h^2}{8mL^2}, \tag{15}$$

which are the same eigenvalues as for a particle in a box (see Eqn. 1 above).