Group Problems #23 - Solutions

Monday, October 24

Problem 1 *Practice with a classical problem*

A mass m is dropped from rest at a height H above a tank of water. On entering the water, it is subject to a buoyant force B greater than the weight of the object. (We neglect the viscous force exerted by the water.) Find the displacement and velocity of the object from the time it is released until it rises to the surface of the water. **Since this is a classical problem, it is probably easier to start with Newton's 2nd law, $F = ma$, rather than a potential energy, and use the general expression for acceleration, $a_y = d^2y/dt^2$, to generate a differential equation.**

Choose a coordinate system with $+y$ upward and take $y = 0$ at the water surface. During the time the object is initially in free fall, it is subject only to the force of gravity. In this region (region 1: above the water), Newton's 2nd Law gives:

$$F = ma \rightarrow -mg = m\frac{d^2y_1}{dt^2}, \tag{1}$$

where the subscript on y denotes that this is for region 1. This differential equation is trivial to solve:

$$v_1(t) = v_{0_1} - gt \tag{2}$$

$$y_1(t) = y_{0_1} + v_{0_1}t - \frac{1}{2}gt^2. \tag{3}$$

Now applying the initial conditions, $y_{0_1} = H$ and $v_{0_1} = 0$, we get the motion in region 1:

$$v_1(t) = -gt \tag{4}$$

$$y_1(t) = H - \frac{1}{2}gt^2. \tag{5}$$

When the mass enters the water (region 2), the force becomes $B - mg$, so Newton's 2nd Law gives:

$$B - mg = m\frac{d^2y_2}{dt^2}, \tag{6}$$

which has solutions:

$$v_2(t) = v_{0_2} + \left(\frac{B}{m} - g\right) t \tag{7}$$

$$y_2(t) = y_{0_2} + v_{0_2} t + \frac{1}{2}\left(\frac{B}{m} - g\right) t^2. \tag{8}$$

Now apply the boundary conditions at the water surface: the boundary conditions require that the position and velocity of the mass be continuous across the boundary between air and water. Thus, if we take t_1 to be the time the mass enters the water, then:

$$y_1(t_1) = y_2(t_1) \tag{9}$$

$$v_1(t_1) = v_2(t_1). \tag{10}$$

The first condition states that the object does not disappear at one instant and reappear at a different point in space at the next instant. The second condition is equivalent to requiring that the speed changes smoothly at the water's surface. To apply the boundary conditions, we must first find t_1, which is obtained by finding the time at which $y_1 = 0$:

$$y_1(t_1) = H - \frac{1}{2} g t_1^2 = 0 \rightarrow t_1 = \sqrt{\frac{2H}{g}}. \tag{11}$$

We can now find the speed at which the object enters the water, $v_1(t_1)$:

$$v_1(t_1) = -g t_1 = -g\sqrt{\frac{2H}{g}} = -\sqrt{2gH}. \tag{12}$$

The boundary conditions then give:

$$y_2(t_1) = y_{0_2} + v_{0_2}\sqrt{\frac{2H}{g}} + \frac{1}{2}\left(\frac{B}{m} - g\right)\left(\frac{2H}{g}\right) = 0 \tag{13}$$

$$v_2(t_1) = v_{0_2} + \left(\frac{B}{m} - g\right)\sqrt{\frac{2H}{g}} = -\sqrt{2gH}. \tag{14}$$

These two equations may be solved simultaneously for y_{0_2} and v_{0_2}, yielding $v_{0_2} = -(B/m)\sqrt{2H/g}$ and $y_{0_2} = H(1 + B/mg)$. The complete solutions in region 2 are thus:

$$v_2(t) = -\frac{B}{m}\sqrt{\frac{2H}{g}} + \left(\frac{B}{m} - g\right) t \tag{15}$$

$$y_2(t) = H + \frac{HB}{mg} - \frac{B}{m}\sqrt{\frac{2H}{g}} t + \frac{1}{2}\left(\frac{B}{m} - g\right) t^2. \tag{16}$$

The equations for v_1, y_1, v_2, and y_2 give the behavior of the object from $t = 0$ until it rises to the surface of the water.