Group Problems #19 - Solutions

Friday, October 7

Problem 1 *Time evolution of position uncertainty*

A particle is initially moving along the x-direction with velocity $v \ll c$. At $t = 0$, we measure the position of a particle, and find it to be somewhere in the interval $-a/2 \leq x \leq a/2$. Where will we be likely to find the particle at times $t > 0$?
At $t = 0$, the particle can be found anywhere in the given range, so $\Delta x = a$. By the uncertainty principle, this implies that $\Delta p \sim \hbar/a$, or $\Delta v \sim \hbar/ma$, where m is the particle's mass. The best estimate of the particle's location at $t = 0$ is in the center of the given range, so its position is $x = 0 \pm a/2$ and its velocity is $v \pm \hbar/2ma$.

As time evolves, the uncertainty in the particle's position changes due to two mechanisms: 1) the initial uncertainty in its position as given above, and 2) the uncertainty in its initial velocity leads to a range of possible positions at a later time. To see how this works, we first must realize that if there were no uncertainty in the initial position and velocity, the particle's trajectory would be exactly determined for all future times: $x(t) = 0 + vt = vt$.

Now we artificially set $\Delta v = 0$ (at $t = 0$) to investigate the first effect described above. Under this assumption, we have:

$$vt - \frac{a}{2} \leq x(t) \leq vt + \frac{a}{2} \Longrightarrow \Delta x_a(t) = a, \tag{1}$$

where the subscript a in the last expression indicates that this is the position uncertainty due to the initial constraint on the particle's position. Now we artificially set $\Delta x = 0$ (at $t = 0$) to investigate the second effect described above. Under this assumption, we have:

$$v - \frac{\hbar}{2ma} \leq \quad v \quad \leq v + \frac{\hbar}{2ma}, \text{ so} \tag{2}$$

$$(v - \frac{\hbar}{2ma})t \leq \quad x(t) \quad \leq (v + \frac{\hbar}{2ma})t \Longrightarrow \Delta x_v(t) = \frac{\hbar}{ma}t, \tag{3}$$

where the subscript v in the last expression indicates that this is the position uncertainty due to the uncertainty in the particle's velocity.

Since these two uncertainties (Δx_a and Δx_v) are uncorrelated, we add them in quadrature, to get the total uncertainty in the position:

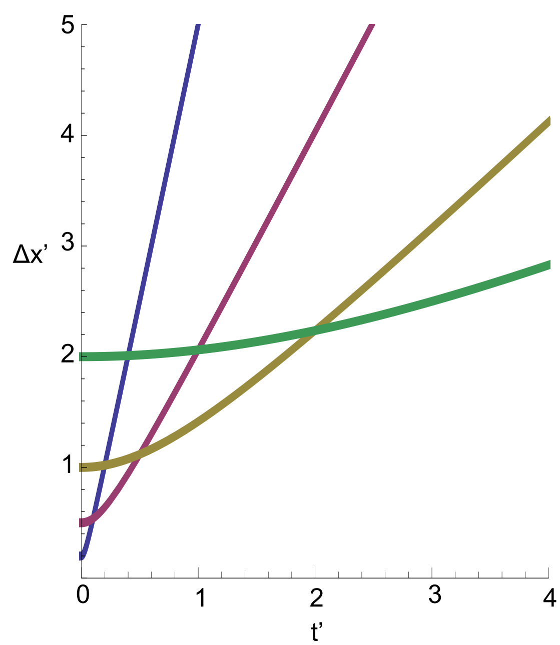
$$\Delta x(t) = \sqrt{(\Delta x_a)^2 + (\Delta x_v)^2} = a\sqrt{1 + \left(\frac{\hbar t}{ma^2}\right)^2}. \qquad (4)$$

From this equation, we can deduce the behavior of $\Delta x(t)$ in the limit of short and long times. For short times, namely when $t \ll ma^2/\hbar$, then we can neglect the second term in the radical and $\Delta x(t) = a$. This makes sense from the discussion above. For long times, namely when $t \gg ma^2/\hbar$, then we can neglect the first term in the radical, and the position uncertainty is linearly dependent on time, $\Delta x(t) = (\hbar/ma)t$. This also makes sense.

The crossover between these two regimes happens at $t = ma^2/\hbar$, for which $\Delta x(t) = \sqrt{2}a$. We see that the crossover happens for shorter and shorter times if the particle is constrained more tightly at $t = 0$, in other words as a becomes smaller. Also notice that when a becomes smaller, the uncertainty increases more rapidly with time. The temporal evolution of Δx is most easily appreciated by plotting $\Delta x(t)$ vs. t for various values of a. To facilitate this, we first divide both sides of the equation above by $\hbar/mc$, which has units of length. This removes the dependence on m and yields dimensionless lengths $\Delta x' = \Delta x/(\hbar/mc)$, $a' = a/(\hbar/mc)$, and dimensionless time $t' = t//(\hbar/mc^2)$. With these definitions, the equation above simplifies to:

$$\Delta x' = \sqrt{(a')^2 + \left(\frac{t'}{a'}\right)^2}. \qquad (5)$$

This is plotted below for four different values of a': 0.2, 0.5, 1, and 2.



Problem 2 *Quantum uncertainty of a classical particle*

The Lord decided to take a break during the creation of the universe 13 billion years ago and play a little billiards. She pockets all the balls (after all, who can she play against?) and leaves the cue-ball ($m = 0.2$ kg) in the center of the table, which measures 1 m on a side. Just today, she decides to take another break (she's been pretty busy in the meantime) and returns to the table only to find the cue ball has moved. How far? (Ignore expansion of the space-time continuum, i.e., the Big Bang.) First, let's determine whether we're in the long-time limit for this problem using the above relations. Here we have $a = 1$ m, $m = 0.2$ kg, so $ma^2/\hbar = (0.2 \text{ kg})(1 \text{ m}^2/(1.05\times 10^{-34} \text{ kg m}^2/\text{s}) \sim 2\times 10^{33}$ s. Thirteen billion years corresponds to $(13\times 10^9 \text{ years})(31.6\times 10^6 \text{ s/year}) = 4.1 \times 10^{17}$ s. So clearly, we are in the short time regime and we should not expect the ball to move very much. To calculate how much it moved, use the uncertainty relation:

$$\Delta x \Delta p \sim \hbar \Longrightarrow \Delta v \sim \frac{\hbar}{m\Delta x} \quad (6)$$

$$= \frac{1.05 \times 10^{-34} \text{ kg m}^2/\text{s}}{(0.2 \text{ kg})(1 \text{ m})} \quad (7)$$

$$= 5.25 \times 10^{-34} \text{ m/s}. \quad (8)$$

If we assume that the cue ball has half this velocity, then in 13 billion years, it will move a distance $d = vt = (2.6 \times 10^{-34} \text{ m/s})(4.1 \times 10^{17} \text{ s}) = 1.08 \times 10^{-16}$ m. So the cue ball has moved a distance corresponding to 1 millionth of a hydrogen atom! The point is, the position of macroscopic objects is known with very small uncertainty.