Group Problems #18 - Solutions

Wednesday, October 5

Problem 1 *Classical position-momentum uncertainty*

In a measurement of the wavelength of water waves, 10 wave crests are counted in a distance of 200 cm. Estimate the minimum uncertainty in the wavelength that might be obtained from this experiment.

Here, we should use the classical position-momentum uncertainty relation for waves: $\Delta x \Delta k \geq \pi$, where Δx is the uncertainty related to the position of the wave, and Δk is the uncertainty associated with its wavenumber ($k = 2\pi/\lambda$). First, since the problem asks about the wavelength, we should convert $\Delta k \to \Delta\lambda$:

$$k = \frac{2\pi}{\lambda} \Longrightarrow Deltak = -\frac{2\pi}{\lambda^2}\Delta\lambda, \tag{1}$$

where we have just taken the derivative (actually the differential) of the expression for k with respect to λ. We don't care about the "$-$" sign, so now we have:

$$\Delta x \left(\frac{2\pi}{\lambda^2}\right) \Delta\lambda \geq \pi \Longrightarrow \Delta\lambda \geq \frac{\lambda^2}{2\Delta x}. \tag{2}$$

The best estimate for λ is just the distance over which the measurement is obtained (200 cm) divided by the number of crests observed (10): $\lambda \sim 200/10 = 20$ cm. The distance over which the measurement is made is also the spread (uncertainty) in x: $\Delta x = 200$ cm. Now we compute:

$$\Delta\lambda \geq \frac{(20\text{ cm})^2}{2(200\text{ cm})} = \frac{400}{400}\text{ cm} = 1\text{ cm}. \tag{3}$$

Does this make sense? Well imagine taking a picture of the wave using a camera whose magnification (zoom) is set so that the full width of the image corresponds to 200 cm at the location of the wave. In the picture, you count the number of wave crests to be 10. (This is just a restatement of the problem.) Because you can only count crests and troughs, you are limited in your ability to determine what fraction of a wave is left on either side of the 10 counted crests near the camera edges. In particular, we could estimate that the fraction to the left and right could each be up to about 1/4 of a wave (any more than that would be fairly easy to see), so we could have as much as 1/2 of an additional wave in our field-of-view. So we might have

10.5 waves in our field-of-view instead of the original estimate of 10, which would give $\lambda = 200/10.5 = 19.05$ cm. Comparing this to our original measurement of $\lambda = 20$ cm, we see that $\Delta\lambda \sim 1$ cm, in agreement with our calculation.

Problem 2 *Classical time-frequency uncertainty*

An electronic salesman offers to sell you a frequency-measuring device. When hooked up to a sinusoidal signal, it automatically displays the frequency of the signal, and to account for frequency variations, the frequency is averaged over a period of 1 second (the display is refreshed each second). The salesman claims the device to be accurate to 0.01 Hz. Is this claim valid?

Again, we use the classical uncertainty relations, this time the one that relates frequency and time: $\Delta\omega\Delta t \geq \pi$, where ω is the angular frequency (in radians/s) of the signal ($\omega = 2\pi\nu$ and ν is the frequency in Hz) and t is time. The duration of each measurement is 1 second, so $\Delta t = 1$ s. Converting the ω to ν and applying the uncertainty principle gives:

$$\Delta\omega\Delta t = 2\pi\Delta\nu\Delta t \geq \pi \Longrightarrow \Delta\nu \geq \frac{1}{2\Delta t} = \frac{1}{2\ \text{s}} = \frac{1}{2}\ \text{Hz}. \tag{4}$$

So the instrument cannot measure to an accuracy better than 0.5 Hz, and the salesman is lying.