Group Problems #14 - Solutions

Monday, September 26

Problem 1 *Connecting Planck's and Wien's Laws*

Show that Planck's Law,

$$\frac{dP}{dA} = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}. \tag{1}$$

can be used to derive Wien's Law, $\lambda_{max} = b/T$, where $b = 2.898 \times 10^{-3}$ m·K.

Hints:

1. In Wien's law, what does λ_{max} signify?
2. It is difficult to find a simple expression in a form that resembles Wien's law. However, you can obtain an expression of the form:

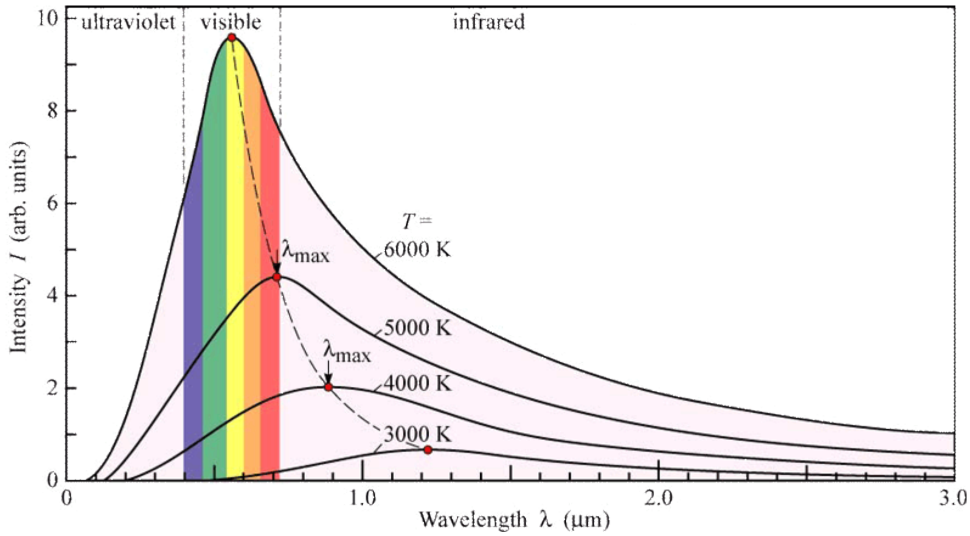
$$5 = (5 - x)e^x \tag{2}$$

   where x is a trivial function of λ and T. This expression is a transcendental equation, which can be solved either graphically or numerically for x.

   Note: there are two solutions to this transcendental equation. The trivial solution, $x = 0$, is not of interest. Why?

Planck's law describes the variation of the power per unit area (intensity, I) emitted from a thermal source (black body) as a function of the emission wavelength λ and the temperature of the source T. Planck's law originates from a first-principles derivation and agrees very well with observations. Wien's law describes the dependance of the wavelength corresponding to maximum emission intensity, λ_{max}, as a function of the source temperature. Wien's law is empirical; that is, it originates from observations only, not from first principles. Wien's law does not predict the detailed shape of the emission intensity spectrum (its dependence on wavelength), while Planck's law does. The graphical relationship between Planck's law and Wien's law is shown in the figure below, where the solid curves give the emission spectra predicted by Planck's law at different temperatures and the dotted line and red circles trace the wavelength of maximum emission intensity, λ_{max}.

To find Wien's law, we notice that for a particular temperature, the Planck spectrum predicts a maximum intensity I_{max} corresponding to λ_{max}, where the slope of



the Planck spectrum (I vs. λ) is zero. Obviously, then, we should take the derivative of the Planck spectrum and set it equal to zero to find a relationship between $\lambda_{\rm max}$ and T. Let's start by rewriting Planck's law:

$$I = \frac{dP}{dA} = \frac{\beta}{\lambda^5}\frac{1}{e^{\alpha/\lambda T} - 1}, \tag{3}$$

where we have simply defined the constants $\alpha = hc/k$ and $\beta = 2\pi hc^2$. With these definitions, we then have:

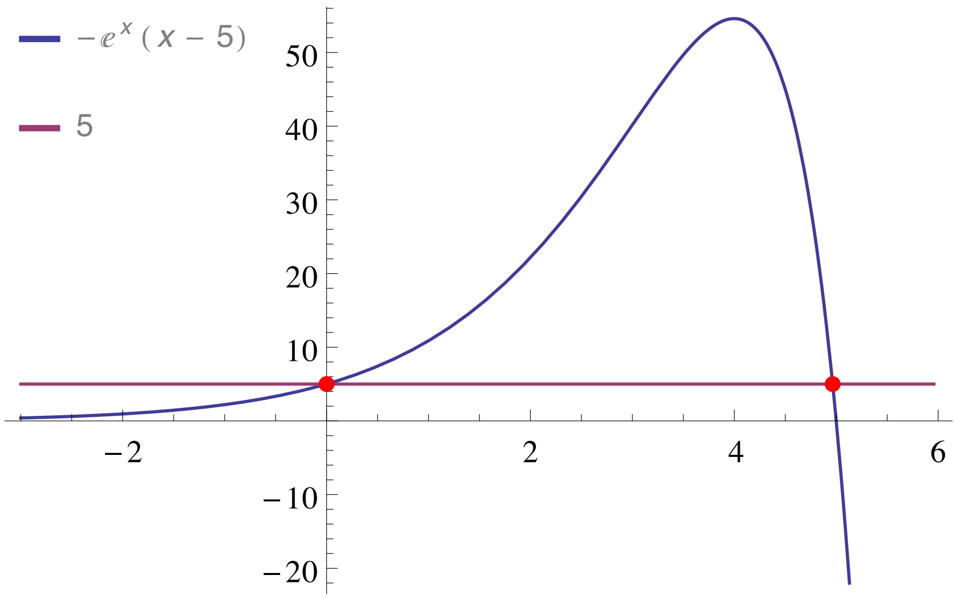
$$\begin{aligned} \frac{dI}{d\lambda} = 0 &= -\frac{5\beta}{\lambda^6}\left[\frac{1}{e^{\alpha/\lambda T} - 1}\right] + \frac{\beta}{\lambda^5}\cdot\frac{-1}{\left(e^{\alpha/\lambda T} - 1\right)^2}\cdot e^{\alpha/\lambda T}\cdot\frac{-\alpha}{\lambda^2 T} && (4)\\ &= \frac{\beta}{\lambda^6\left(e^{\alpha/\lambda T} - 1\right)^2}\left[5 - 5e^{\alpha/\lambda T} + \frac{\alpha}{\lambda T}e^{\alpha/\lambda T}\right]. && (5)\end{aligned}$$

The right hand side of the above equation must be equal to zero. Since $\beta \neq 0$, then the expression in brackets must be null:

$$\begin{aligned} 0 &= 5 - 5e^{\alpha/\lambda T} + \frac{\alpha}{\lambda T}e^{\alpha/\lambda T} && (6)\\ \Longrightarrow 5 &= \left(5 - \frac{\alpha}{\lambda T}\right)e^{\alpha/\lambda T} = (5 - x)e^x, && (7)\end{aligned}$$

where we have made the substitution $x = \alpha/(\lambda T) = hc/(\lambda k T)$. The solution to this transcendental equation will yield the value of λ corresponding to $I_{\rm max}$; that is, it will give $\lambda_{\rm max}$.

As described in the problem hints above, this transcendental equation has two solutions. The trivial solution, $x = hc/(\lambda k T) = 0$, is not a general one since it is only valid in the high-temperature limit $T \to \infty$. The general solution can be obtained graphically by plotting the two equations $y = 5$ and $y = (5 - x)e^x$ on the same graph and looking for the (non-zero) value of x where the two plots intersect. This is shown in the figure below. This plot was produced very simply by using Wolfram Alpha



(http://www.wolframalpha.com/); I simply typed the following string into the search box: **plot (5 - x)*e^x = 5**. If you have never used Wolfram Alpha before, you should become familiar with it: it is an incredibly useful tool!

The graph shows that the non-trivial solution corresponds to $x = hc/(\lambda kT) \simeq 5$ (the actual value is $x = 4.96511$). Rearranging this, and using $hc = 1240$ eV-nm and $k = 8.62 \times 10^{-5}$ eV/K (the Boltzmann constant), we find:

$$\lambda_{\max} = \frac{hc}{4.97\,kT} = \frac{b}{T}, \tag{8}$$

where $b = hc/(4.97\,k) = \frac{1240}{(4.97)(8.62)} \times 10^5$ nm-K $= 2.89 \times 10^{-3}$ m-K, which reproduces Wien's law.