Group Problems #13 - Solutions

Wednesday, September 21

Problem 1 *Blackbody Radiation*

The filament of a light bulb is cylindrical with length $l = 20$ mm and radius $r = 0.05$ mm. The filament is maintained at a temperature $T = 5000$ K by an electric current. The filament behaves approximately as a black body, emitting radiation isotropically. At night, you observe the light bulb from a distance $D = 10$ km with the pupil of your eye fully dilated to a radius $\rho = 3$ mm.

(a) What is the total power radiated by the filament?

The Stefan-Boltzman Law is:

$$\frac{dP}{dA} = \sigma T^4, \tag{1}$$

where dP/dA is the power radiated per unit surface area of the source, $\sigma = 5.6 \times 10^{-8}$ W$\cdot$m$^{-2}\cdot$K^{-4} is the Stefan-Boltzman constant, and T is the temperature in Kelvin. Multiplying both sides of the equation by dA and integrating over the surface area of the filament, gives:

$$\begin{aligned} P_{\text{tot}} &= A\sigma T^4 = 2\pi r l \sigma T^4 && (2)\\ &= 2\pi(0.05 \times 10^{-3}\,\text{m})(0.02\,\text{m})(5.6 \times 10^{-8}\,\text{W}\cdot\text{m}^{-2}\cdot\text{K}^{-4})(5000\,\text{K})^4 && (3)\\ &= 220 \text{ W}. && (4) \end{aligned}$$

(b) How much radiation power enters your eye?

Since the lightbulb emits radiation equally in all directions (isotropic), the power entering your eye is just the total power radiated times the fraction of your pupil disk to the total surface area of a sphere centered at the source and of radius $D = 10$ km. Thus, we have:

$$P_{\text{eye}} = P_{\text{tot}} \frac{\pi(3 \times 10^{-3}\,\text{m})^2}{4\pi(10 \times 10^3\,\text{m})^2} = 4.95 \times 10^{-12}\,\text{W}. \tag{5}$$

(c) At what wavelength does the filament radiate the most power?

Here, we simply use Wien's Law:

$$\lambda_{\text{max}} = \frac{b}{T} = \frac{2.9 \times 10^{-3}\,\text{m} \cdot \text{K}}{5000\,\text{K}} = 5.8 \times 10^{-7}\,\text{m} = 580\ \text{nm}. \tag{6}$$

(d) How many radiated photons enter your eye every second? You can assume that the average wavelength for the radiation is $\bar{\lambda} = 600$ nm.

The energy of a photon is given by $E_{\text{photon}} = hc/\lambda$. So, dividing the total power (energy per unit time) entering your eye by the energy of an average photon, we get:

$$N_{\text{photons}} = \frac{P_{\text{eye}}}{hc/\bar{\lambda}} = \frac{(4.95 \times 10^{-12}\,\text{W})(600 \times 10^{-9}\,\text{m})}{(6.63 \times 10^{-34}\,\text{J} - \text{s})(3 \times 10^{8}\,\text{m/s})} \tag{7}$$

$$= 1.5 \times 10^{7}\,\text{photons/s}. \tag{8}$$

Note that the most sensitive electronic photon detectors have a quantum efficiency near 100% in the visible spectral region; that is, a measurable electronic signal is produced for nearly every photon incident on the detector. They also have quantum noise due to thermal energy (dark noise) of $\sim$100 photons/s at room temperature. Thus, a signal of $\sim 10^7$ photons/s would be easily measured. The human eye has a quantum efficiency of around 1%, and a dark noise that depends on how long the eye has been in the dark. However, this signal would probably still be detectable by the human eye.

Problem 2 *Stefan-Boltzmann Law*

If we consider the Earth as a blackbody in thermal equilibrium,

(a) estimate the global temperature of our planet in terms of the temperature of the Sun, T_{sun}, its radius R_{sun}, and distance D between the Earth and Sun.

Thermal equilibrium implies as much power is radiated by the earth into space as is received from the sun, so

$$P_{\text{in}} = P_{\text{out}}, \tag{9}$$

where P_{in} is the power received by the sun and P_{out} is the power radiated by the earth. The *total* power radiated by the sun is the power density (power per unit area in this case), denoted by dP_{sun}/dA, integrated over the entire surface area of the sun, $4\pi R_{\text{sun}}^2$. At a distance D from the sun, the earth only receives a fraction of this power:

$$P_{\text{in}} = \frac{dP_{\text{sun}}}{dA}\,4\pi R_{\text{sun}}^2\,\frac{\pi R_{\text{earth}}^2}{4\pi D^2} \tag{10}$$

We can now use the Stefan-Boltzmann law to substitute for $dP_{\rm sun}/dA = \sigma\, T_{\rm sun}^4$, where $\sigma = 5.67 \times 10^{-8}$ W/m^2/K^4 is Stefan's constant. Substituting into the Eqn. above gives:

$$P_{\rm in} = \sigma\, T_{\rm sun}^4\, 4\pi R_{\rm sun}^2\, \frac{\pi R_{\rm earth}^2}{4\pi D^2} \tag{11}$$

Now considering the Earth also as a blackbody, we can use a similar procedure to find $P_{\rm out}$. The total power radiated by the Earth is the power density integrated over the surface area of the Earth:

$$\begin{aligned} P_{\rm out} &= \frac{dP_{\rm earth}}{dA}\, 4\pi R_{\rm earth}^2 && (12) \\ &= \sigma\, T_{\rm earth}^4\, 4\pi R_{\rm earth}^2, && (13) \end{aligned}$$

where we have again used the Stefan-Boltzmann law in the second line of the equation above. Equating the input and output powers and simplifying gives:

$$\begin{aligned} P_{\rm in} = P_{\rm out} \quad &\longrightarrow \quad \sigma\, T_{\rm sun}^4\, 4\pi R_{\rm sun}^2\, \frac{\pi R_{\rm earth}^2}{4\pi D^2} = \sigma\, T_{\rm earth}^4\, 4\pi R_{\rm earth}^2 && (14) \\ &\longrightarrow \quad T_{\rm earth}^4 = \frac{R_{\rm sun}^2\, T_{\rm sun}^4}{4D^2} && (15) \\ &\longrightarrow \quad T_{\rm earth} = T_{\rm sun}\sqrt{\frac{R_{\rm sun}}{2D}} && (16) \end{aligned}$$

(b) Compute this temperature using $T_{\rm sun} = 5700$ K, $R_{\rm sun} = 7 \times 10^5$ km, and $D = 150 \times 10^6$ km.

Plugging numbers gives:

$$\begin{aligned} T_{\rm earth} &= 5700\sqrt{\frac{7 \times 10^5}{2(150 \times 10^6)}} = 5700\sqrt{7/3 \times 10^{-3}} && (17) \\ &= 275 \text{ K} \simeq 2\ ^\circ\text{C}. && (18) \end{aligned}$$

This is actually a reasonable number since it is above freezing - otherwise we would expect more ocean water to be locked up in ice. Recent data suggests that the Earth's temperature is trending upward faster than previous natural fluctuations. The strongly supported hypothesis is that this warming is caused by the greenhouse effect (which is not in dispute, as it is clearly understood in many systems across many scales), which tends to trap a portion of Earth's outgoing radiation, thus reducing $P_{\rm out}$ without effecting $P_{\rm in}$. This would clearly lead to a non-equilibrium situation that results in warming of the globe.