Group Problems #12 - Solutions

Monday, September 19

Problem 1 *Energy-momentum invariant*

(a) What is the value of the energy-momentum invariant for a particle of mass m?

The modulus of a particle's 4-momentum is defined as $\mathbf{P}^2 = \mathbf{P} \cdot \mathbf{P} = (E/c)^2 - p^2$. This quantity is invariant with respect to a change in reference frames, so to compute its value we are free to choose a convenient frame. Choosing the frame in which the particle is at rest gives $p = 0$ and $E = \gamma mc^2 = mc^2$, since $v = 0$ and $\gamma = 1$ in this frame. So we have $(E/c)^2 - p^2 = (mc^2/c)^2 = m^2c^2$. Traditionally, we define the "energy-momentum invariant" in energy units, so we multiply through by c^2:

$$E^2 - (pc)^2 = m^2c^4. \tag{1}$$

So the "value" of the energy-momentum invariant is m^2c^4. It is also OK to use momentum units, in which case the value is m^2c^2.

(b) What is the value of the energy-momentum invariant for a massless particle?

From above we see that $E^2 - (pc)^2 = 0$ for a massless ($m = 0$) particle. Thus, the value of the energy-momentum invariant is zero.

Problem 2 *Compton scattering*

A beam of x-rays with wavelength 0.2400 nm is directed toward a sample. The x-rays scatter from the electrons within the sample, imparting momentum to the electrons, which are initially at rest in the lab frame. After scattering, the x-rays are detected at various angles relative to the direction of the incoming beam using a detector that can resolve their wavelengths.

(a) What is the longest wavelength measured by the detector?

The Compton scattering formula derived in the lecture notes is:

$$\lambda' - \lambda = \frac{hc}{mc^2}(1 - \cos\theta), \tag{2}$$

where λ is the initial x-ray wavelength, λ' is the scattered x-ray wavelength, m is the mass of the recoiling particle (in this case an electron: $mc^2 = 511$ keV), and θ is the scattering angle of the x-rays relative to the initial beam direction. Note that $\lambda' \geq \lambda$ since $-1 \leq \cos\theta \leq 1$. This makes sense since the scattering process imparts energy and momentum to the recoiling electron, and thus the x-ray energy must decrease (the wavelength must get longer).

The largest shift in the wavelength occurs for $\cos\theta = -1$. In this case,

$$\lambda' = \lambda + 2\frac{hc}{mc^2}. \tag{3}$$

A very useful quantity to remember is Planck's constant times the speed of light: $hc = 1240$ eV-nm. With this, we find:

$$\lambda' = 0.2400 \text{ nm} + 2\,\frac{1240 \text{ eV} - \text{nm}}{511 \times 10^3 \text{ eV}} = 0.2400 + 0.0049 = 0.2449 \text{ nm}. \tag{4}$$

(b) At what scattering angle does this occur?

$\cos\theta = -1 \longrightarrow \theta = \pi$. The x-ray scatters directly backwards, which gives the biggest momentum kick to the electron. This corresponds to the largest possible loss of energy from the x-ray photon.

(c) For this scattering angle, what is the *kinetic energy* of the recoiling electrons?

Conserving the first component of the 4-momentum before and after the scattering process is equivalent to conservation of energy. So we have:

$$E + m_e c^2 = E' + \gamma m_e c^2 \Longrightarrow E - E' = (\gamma - 1)m_e c^2, \tag{5}$$

where E is the initial energy of the x-ray photon and E' is the energy of the scattered x-ray. We recognize that the last term in the equation, $(\gamma - 1)m_e c^2$ is the kinetic energy of the recoiling electron. Thus, we must convert the initial and scattered x-ray wavelengths into energy using $E = hc/\lambda = 1240$ eV-nm$/\lambda$.

For the current problem, we have $E = 1240/0.24 = 5167$ eV, and $E' = 1240/0.2449 = 5063$ eV, so $E - E' = 104$ eV, which is the kinetic energy of the recoiling electron. Note that this is much smaller than the rest energy of an electron (511 keV), so the recoiling electron is non-relativistic.

(d) If the detector measures a wavelength for the scattered x-rays of 0.2412 nm, what is the x-ray scattering angle?

Rearranging the Compton scattering formula (Eq. 2) gives:

$$\begin{aligned}
\cos\theta &= (\lambda - \lambda')\,\frac{m_e c^2}{hc} + 1 && (6)\\
&= -0.0012\frac{511 \times 10^3}{1240} + 1 && (7)\\
&= 0.5 && (8)\\
&\longrightarrow \theta = \pi/3. && (9)
\end{aligned}$$

(e) What is the direction of travel of the recoiling electrons in this case?

The energy of the scattered x-ray photon is $E' = hc/\lambda' = 1240/0.2412 = 5141$ eV, so its momentum is $p' = E'/c = 5141$ eV$/c$. It travels at a 60° angle, so $p'_x = 5141\cos(\pi/3) = 2570.5$ eV$/c$ and $p'_y = 5141\sin(\pi/3) = 4452.3$ eV$/c$. Conserving the x- and y-components of momentum before and after the scattering process then gives:

$$x - \text{direction}: \; p_{i,\text{tot}} = p_{f,\text{tot}} \longrightarrow 5167 \text{ eV}/c = 2570.5 \text{ eV}/c + p'_{e,x} \quad (10)$$

$$\longrightarrow p'_{e,x} = 2596.5 \text{ eV}/c \quad (11)$$

$$y - \text{direction}: \; p_{i,\text{tot}} = p_{f,\text{tot}} \longrightarrow 0 = 4452.3 \text{ eV}/c + p'_{e,y} \quad (12)$$

$$\longrightarrow p'_{e,y} = -4452.3 \text{ eV}/c \quad (13)$$

where $p_{i,\text{tot}}$ and $p_{f,\text{tot}}$ are the total initial and final momentum in each direction, and $p'_{e,x}$ and $p'_{e,y}$ is the momentum of the recoiling electron in the x- and y-directions, respectively. With this information, we can solve for the direction of travel, ϕ, for the recoiling electrons:

$$\tan\phi = \frac{p'_{e,y}}{p'_{e,x}} = -\frac{4452.3}{2596.5} \longrightarrow \phi = 59.75°. \quad (14)$$