Group Problems #11 - Solutions

Friday, September 16

Problem 1 *A relativistic collision*

A muon at rest and with mass $m_\mu = 106$ MeV$/c^2$, interacts with an incoming neutrino, of energy $E_\nu = 5$ MeV and negligible mass. After the interaction, the emerging particles are an electron (mass $m_e = 0.511$ MeV$/c^2$) and a neutrino. The direction of the outgoing neutrino makes an angle θ with that of the incoming one.

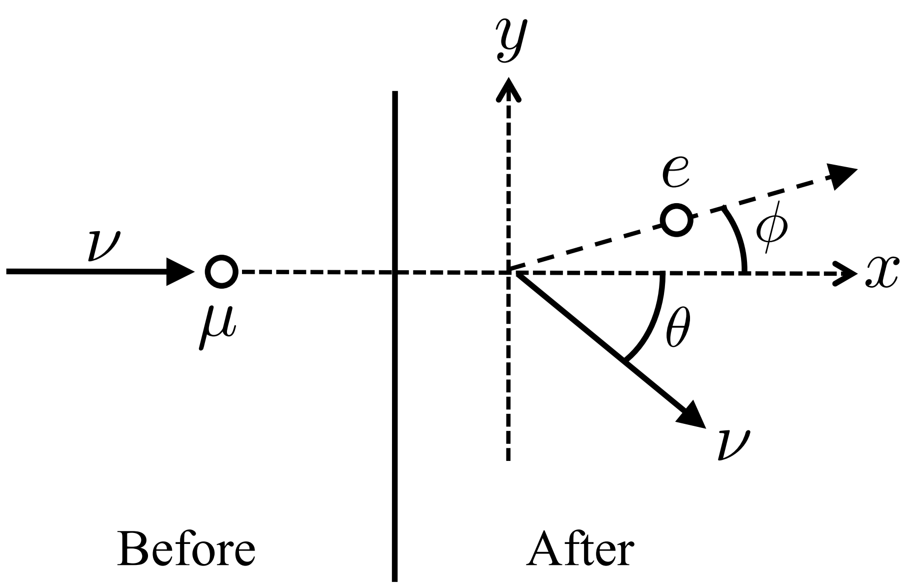

Figure 1: Scattering Geometry.

(a) Choose an appropriate reference frame and write down the energy-momentum invariant $(E/c)^2 - p^2$ of the system before the reaction occurs (p and E are the total 3-momentum and energy before the reaction).

Given a particle of energy E and three-momentum $\mathbf{p}$, the energy-momentum four-vector is $P = (E/c, \mathbf{p})$. For this specific problem, the four-momentum of the muon is

$$P_\mu = (m_\mu c, 0, 0, 0), \tag{1}$$

since the muon is at rest in its own frame: the three-momentum is zero and the energy is just its rest energy $m_\mu c^2$. The incoming neutrino, which we choose to travel in the $+x$ direction, has negligible mass and energy $E_\nu = 5$ MeV. To find the magnitude of the neutrino momentum p_ν, we use the energy-momentum invariant $E_\nu^2 - p_\nu^2 c^2 = m_\nu^2 c^4$, which is $= 0$ since the neutrino has zero mass. The magnitude of the neutrino momentum is thus $p_\nu = E_\nu/c = 5$ MeV$/c$. For a particle of mass zero, the modulus of its momentum is always equal to its energy over c. The neutrino then has four-momentum

$$P_\nu = (E_\nu/c, E_\nu/c, 0, 0). \tag{2}$$

The four-momentum of the initial state (muon plus neutrino) is then

$$P_{\text{initial}} = P_\mu + P_\nu = (m_\mu c + E_\nu/c, E_\nu/c, 0, 0). \tag{3}$$

Since P_{initial} is the sum of two 4-vectors, it is also a 4-vector and it must therefore have an associated invariant, or modulus; the energy-momentum invariant of the initial system is thus:

$$P_{\text{initial}}^2 = (m_\mu c + E_\nu/c)^2 - (E_\nu/c)^2 = m_\mu^2 c^2 + 2m_\mu E_\nu. \tag{4}$$

(b) Write down the energy-momentum invariant after the reaction has occurred.

In the final state, there is an electron and a neutrino. The electron moves at an unknown angle ϕ with respect to the direction of the incoming neutrino, so its 4-momentum is

$$P_e = (E_e/c, p_e \cos\phi, p_e \sin\phi, 0). \tag{5}$$

The energy of the outgoing neutrino is E'_ν, and the magnitude of its 3-momentum is $p'_\nu = E'_\nu/c$, so its 4-momentum is:

$$P'_\nu = (E'_\nu/c, (E'_\nu/c)\cos\theta, -(E'_\nu/c)\sin\theta, 0). \tag{6}$$

The total 4-momentum of the final state is the sum of that for the electron and neutrino:

$$P_{\text{final}} = P_e + P'_\nu = ((E_e + E'_\nu)/c, (E'_\nu/c)\cos\theta + p_e\cos\phi, -(E'_\nu/c)\sin\theta + p_e \sin\phi, 0). \tag{7}$$

Before computing P_{final}^2, we can simplify the expression for P_{final}. Starting from $P_{\text{initial}} = P_{\text{final}}$, each component must be conserved separately. Conserving the $x-$component of the 3-momentum gives:

$$E_\nu/c = (E'_\nu/c)\cos\theta + p_e \cos\phi. \tag{8}$$

Conserving the $y-$component gives:

$$0 = -(E'_\nu/c)\sin\theta + p_e \sin\phi. \tag{9}$$

So $P_{\text{final}} = ((E_e + E'_\nu)/c, E_\nu/c, 0, 0)$, and:

$$P_{\text{final}}^2 = (E_e + E'_\nu)^2/c^2 - (E_\nu/c)^2. \tag{10}$$

(c) Using the equality of this invariant before and after the reaction, find an expression for the momentum of the electron p_e, assuming that $\theta = 0$.

Because $P_{\text{final}} = P_{\text{initial}}$ then the associated invariants are also equal, $P_{\text{final}}^2 = P_{\text{initial}}^2$:

$$m_\mu^2 c^2 + 2m_\mu E_\nu = (E_e + E'_\nu)^2/c^2 - (E_\nu/c)^2. \tag{11}$$

We have two unknowns in the above equation, E_e and E'_ν. Since we want to solve for the three-momentum p_e, we use the relation $E_e = \sqrt{p_e^2 c^2 + m_e^2 c^4}$ to eliminate E_e and obtain

$$m_\mu^2 c^2 + 2m_\mu E_\nu = (\sqrt{p_e^2 c^2 + m_e^2 c^4} + E'_\nu)^2/c^2 - (E_\nu/c)^2. \tag{12}$$

To get rid of E'_ν, we use again the other equation on the conservation of the three-momentum. We cannot use the conservation of the three-momentum in the x direction immediatly, because this expression contains both E'_ν and ϕ, which is also unknown. So, we first solve for $\sin\phi$ using the conservation of the three-momentum in the y direction, obtaining

$$p_e \sin\phi = (E'_\nu/c)\sin\theta. \tag{13}$$

Using a trigonometric rule, we then write

$$p_e \cos\phi = \sqrt{p_e^2 - (E'_\nu/c)^2 \sin^2\theta}. \tag{14}$$

The conservation of three-momentum in the x direction is then

$$E_\nu/c = (E'_\nu/c)\cos\theta + \sqrt{p_e^2 - (E'_\nu/c)^2 \sin^2\theta}. \tag{15}$$

This last equation can be written as a quadratic equation in E'_ν. In fact, it can be written as

$$E_\nu/c - (E'_\nu/c)\cos\theta = \sqrt{p_e^2 - (E'_\nu/c)^2 \sin^2\theta}, \tag{16}$$

so when squared on both sides gives

$$(E_\nu/c)^2 - 2(E_\nu E'_\nu/c^2)\cos\theta + (E'_\nu/c)^2\cos^2\theta = p_e^2 - (E'_\nu/c)^2 \sin^2\theta, \tag{17}$$

or

$$(E'_\nu/c)^2 - 2(E_\nu E'_\nu/c^2)\cos\theta + (E_\nu/c)^2 - p_e^2 = 0. \tag{18}$$

Solving for E'_ν gives

$$E'_\nu = E_\nu \cos\theta + \sqrt{p_e^2 c^2 - E_\nu^2 \sin^2\theta}. \tag{19}$$

We only kept the "+" solution in the quadratic equation. The invariant interval conservation in Eq. (12) with this expression for E'_ν is

$$m_\mu^2 c^2 + 2m_\mu E_\nu = \frac{\left(\sqrt{p_e^2 c^2 + m_e^2 c^4} + E_\nu \cos\theta + \sqrt{p_e^2 c^2 - E_\nu^2 \sin^2\theta}\right)^2}{c^2} - \frac{E_\nu^2}{c^2}. \tag{20}$$

This equation above can in principle be solved for p_e to obtain the value of the electron momentum at $\theta = 0$.

(d) Find an expression for p_e, if the angle θ is arbitrary.

For arbitrary θ, such an expression is not easy to solve analytically and it's best to solve it numerically. This is generally beyond the scope of this class. However, for completeness, the graph below shows p_e vs. θ. As expected, the largest value of p_e occurs for $\theta = \pi$, which corresponds to the neutrino bouncing straight backward after the collision.