EXERCISE 5.6 In a certain reference frame a static, uniform, electric field E_0 is parallel to the x axis, and a static, uniform, magnetic induction $B_0 = 2E_0$ lies in the $x-y$ plane, making an angle θ with the x axis. Determine the relative velocity of a reference frame in which the electric and magnetic fields are parallel. What are the fields in that frame for $\theta \ll 1$ and $\theta \to (\pi/2)$?

Solution

We define the primed coordinates to be in the reference frame in which the two fields are parallel. The primed frame is moving with velocity v relative to the unprimed frame. We work in units where $c = 1$.

First, let's solve this problem for a general situation where $\vec{E} = (E_0, 0, 0)$ and $\vec{B} = (B_0 \cos\theta, B_0 \sin\theta, 0)$. At the end, we will plug in for $B_0 = 2E_0$. In principle, it is not clear in which direction the primed frame should be moving to achieve the desired result (parallel fields), so let's just use guess and check. If the primed reference frame is moving along the x-axis with relative speed v, then the fields will be transformed in the following way

$$\begin{aligned}
E'_x &= E_x \\
E'_y &= \gamma\left(E_y - vB_z\right) \\
E'_z &= \gamma\left(E_z + vB_y\right) \\
B'_x &= B_x \\
B'_y &= \gamma\left(B_y + vE_z\right) \\
B'_z &= \gamma\left(B_z - vE_y\right)
\end{aligned}$$

where γ has its usual meaning. Substituting the appropriate values for E_x, E_y, E_z, B_x, B_y, and B_z into the previous transformations gives

$$\begin{aligned}
E'_x &= E_0 \\
E'_y &= 0 \\
E'_z &= \gamma v B_0 \sin\theta \\
B'_x &= B_0 \cos\theta \\
B'_y &= \gamma B_0 \sin\theta \\
B'_z &= 0.
\end{aligned}$$

The fields described above are obviously not parallel, since $\vec{B}$ has a y component where $\vec{E}$ does not, and $\vec{E}$ has a z component where $\vec{B}$ does not.

Now consider the case where the primed reference frame is moving along y-axis with relative speed v. In this situation, the fields will transform in the following manner

$$\begin{aligned}
E'_x &= \gamma\left(E_x + vB_z\right) \\
E'_y &= E_y \\
E'_z &= \gamma\left(E_z - vB_x\right) \\
B'_x &= \gamma\left(B_x - vE_z\right) \\
B'_y &= B_y \\
B'_z &= \gamma\left(B_z + vE_x\right).
\end{aligned}$$

Substituting in the appropriate values for each component of $\vec{E}$ and $\vec{B}$ gives

$$\begin{aligned}
E'_x &= \gamma E_0 \\
E'_y &= 0 \\
E'_z &= -\gamma v B_0 \cos\theta \\
B'_x &= \gamma B_0 \cos\theta \\
B'_y &= B_0 \sin\theta \\
B'_z &= \gamma v E_0.
\end{aligned}$$

Once again, these fields cannot be parallel since $\vec{B}$ has a y component and $\vec{E}$ does not.

Finally, let's try the primed reference frame moving along the z-axis. For the primed reference frame moving with speed v along the z-axis the fields are given by

$$\begin{aligned}
E'_x &= \gamma\left(E_x - vB_y\right)\\
E'_y &= \gamma\left(E_y + vB_x\right)\\
E'_z &= E_z\\
B'_x &= \gamma\left(B_x + vE_y\right)\\
B'_y &= \gamma\left(B_y - vE_x\right)\\
B'_z &= B_z.
\end{aligned}$$

Substituting in the appropriate values for each component of $\vec{E}$ and $\vec{B}$ once again gives

$$\begin{aligned}
E'_x &= \gamma\left(E_0 - vB_0\sin\theta\right) && (1)\\
E'_y &= \gamma vB_0\cos\theta && (2)\\
E'_z &= 0\\
B'_x &= \gamma B_0\cos\theta && (3)\\
B'_y &= \gamma\left(B_0\sin\theta - vE_0\right) && (4)\\
B'_z &= 0. && (5)
\end{aligned}$$

These fields could be parallel for an appropriate choice of v. Hence we have to assume that the primed reference frame is moving along the z-axis.

If $\vec{E}'$ and $\vec{B}'$ are parallel, then $\vec{E}'\times\vec{B}' = 0$. Therefore, to find v, just set the cross product of $\vec{E}'$ and $\vec{B}'$ equal to zero and solve for v. The cross product will only have a z component, because $E'_z = B'_z = 0$.

$$\vec{E}'\times\vec{B}' = \hat{z}\left(E'_xB'_y - E'_yB'_x\right)$$

Using the values found in (1)-(5) for the primed components and setting the cross product equal to zero, one gets

$$\begin{aligned}
0 &= \gamma^2\left[\left(E_0 - vB_0\sin\theta\right)\left(B_0\sin\theta - vE_0\right) - vB_0^2\cos^2\theta\right]\hat{z}\\
&= \gamma^2\left[v^2E_0B_0\sin\theta - v\left(E_0^2 + B_0^2\right) + E_0B_0\sin\theta\right]\hat{z}\,.
\end{aligned}$$

Assuming $\gamma \neq 0$, just use the quadratic formula to solve for v, yielding

$$v = \frac{E_0^2 + B_0^2 \pm \sqrt{\left(E_0^2 + B_0^2\right)^2 - 4E_0^2B_0^2\sin^2\theta}}{2E_0B_0\sin\theta}. \tag{6}$$

We keep here the minus sign in front of the square root, since the plus sign leads to $v > 1$. Now, let's use that $B_0 = 2E_0$. For this case, the above expression for v simplifies down to

$$v = \frac{5 - \sqrt{25 - 16\sin^2\theta}}{4\sin\theta} \tag{7}$$

as long as E_0 is nonzero.

Now that we have determined the velocity of the primed reference frame, let's move on to describe the fields in this reference frame. The calculation is greatly simplified by taking advantage of the following two invariants,

$$\vec{B}'\cdot\vec{E}' = \vec{B}\cdot\vec{E} \tag{8}$$

and

$${B'_0}^2 - {E'_0}^2 = B_0^2 - E_0^2. \tag{9}$$

Substituting $B_0 = 2E_0$ into (8) and (9) and employing the fact that $\vec{E}'$ and $\vec{B}'$ are parallel reduces the above equations to

$$B'_0E'_0 = 2E_0^2\cos\theta \tag{10}$$

and

$$B_0'^{\,2} - E_0'^{\,2} = 3E_0^2. \tag{11}$$

Solving (11) for B_0' and substituting this into (10) gives

$$E_0' \sqrt{E_0'^{\,2} + 3E_0^2} = 2E_0^2 \cos\theta. \tag{12}$$

Define $\alpha \equiv E_0'^{\,2}$ and simplify, resulting in a quadratic equation for α

$$\alpha^2 + 3\alpha E_0^2 - 4E_0^4 \cos^2\theta = 0. \tag{13}$$

Finally, solve for α using the quadratic formula, and then solve for E_0' using the definition of α.[1] The final answer for E_0' is

$$E_0' = E_0 \sqrt{\frac{-3 + \sqrt{9 + 16\cos^2\theta}}{2}}. \tag{14}$$

To find B_0', follow the same procedure as above, except solve Eq. (11) for $E_0'^{\,2}$ and plug this into Eq. (10). The final answer for B_0' is

$$B_0' = E_0 \sqrt{\frac{3 + \sqrt{9 + 16\cos^2\theta}}{2}}. \tag{15}$$

In the limit where $\theta \to 0$, $\cos\theta \to 1$, and therefore

$$E_0' \to E_0 \sqrt{\frac{-3 + \sqrt{9 + 16}}{2}} = E_0, \tag{16}$$

and

$$B_0' \to E_0 \sqrt{\frac{3 + \sqrt{9 + 16}}{2}} = 2E_0 = B_0. \tag{17}$$

This makes sense, because in this limit, $\vec{B}$ and $\vec{E}$ are already parallel without the need for a change of reference frame. Therefore, the fields in the reference frame that makes $\vec{B}$ and $\vec{E}$ parallel are simply the fields in the unprimed frame.

In the limit where $\theta \to \pi/2$, $\cos\theta \to 0$, and therefore

$$E_0' \to E_0 \sqrt{\frac{-3 + \sqrt{9}}{2}} = 0 \tag{18}$$

and

$$B_0' \to E_0 \sqrt{\frac{3 + \sqrt{9}}{2}} = \sqrt{3}E_0. \tag{19}$$

This also makes sense: if $\vec{E}$ and $\vec{B}$ are perpendicular, the only way that in a reference frame one has $\vec{E}' \times \vec{B}' = 0$ is if one of the primed fields is in fact zero.

---

[1] The quadratic formula gives two solutions, one solution that describes the fields being parallel and one solution that describes the fields being anti-parallel. We consider the + solution.