A1 Vector Algebra and Calculus

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Vector Algebra and Calculus

1. Revision of vector algebra, scalar product, vector product
2. Triple products, multiple products, applications to geometry
3. Differentiation of vector functions, applications to mechanics
4. Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates
5. Vector operators — grad, div and curl
6. Vector Identities, curvilinear co-ordinate systems
7. Gauss' and Stokes' Theorems and extensions
8. **Engineering Applications**

Engineering applications

1. Electricity – Ampère's Law
2. Fluid Mechanics - The Continuity Equation
3. Thermo: The Heat Conduction Equation
4. Mechanics/Electrostatics - Conservative fields
5. The Inverse Square Law of force
6. Gravitational field due to distributed mass
7. Gravitational field inside body
8. Pressure forces in non-uniform flows

1. Electricity – Ampère's Law

If the frequency is low, the displacement current $\partial \mathbf{D}/\partial t$ in Maxwell's equation $\text{curl}\,\mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$ is negligible, and we find

$$\text{curl}\,\mathbf{H} = \mathbf{J}$$

Hence

$$\int_S \text{curl}\,\mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

or

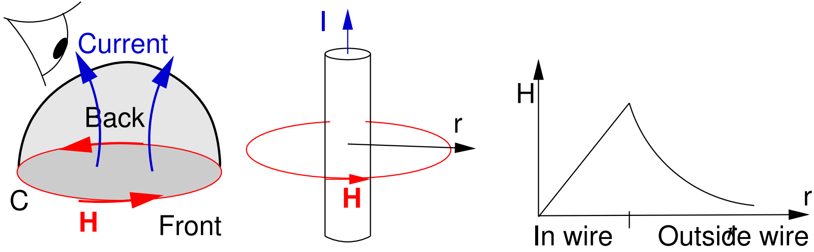
$$\oint_C \mathbf{H} \cdot d\mathbf{r} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

But $\int_S \mathbf{J} \cdot d\mathbf{S}$ is total current I through the surface ...

Electricity – Ampère's Law /ctd

Reminder: $\oint \mathbf{H} \cdot d\mathbf{r} = \int_S \mathbf{J} \cdot d\mathbf{S}$.

Consider wire, radius a carrying current I ...



Inside $r < A$: $\int \mathbf{J} \cdot d\mathbf{S} = I(r^2/a^2) = H2\pi r \Rightarrow \quad H = \left(Ir/2\pi a^2\right)$

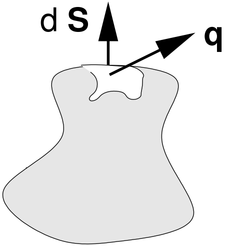
Outside $r > a$: $\int \mathbf{J} \cdot d\mathbf{S} = I = H2\pi \Rightarrow \quad H = (I/2\pi r)$

H is everywhere in the $\hat{\theta}$ direction.

2. Fluid Mechanics - The Continuity Equation 1

The **Continuity Equation** expresses conservation of mass in fluid flow.

Apply this to a **control volume**:
"The net rate of mass flow of fluid out of the control volume must equal the rate of decrease of the mass of fluid within the control volume."



Control Volume V

Fluid Velocity is $\mathbf{q}(\mathbf{r})$ (vector field)
Fluid Density is $\rho(\mathbf{r})$ (scalar field)
Element of rate-of-volume-gain from surface $d\mathbf{S}$:

$$d(\dot{V}) = -\mathbf{q} \cdot d\mathbf{S}$$

$\Rightarrow$the element of rate-of-mass-gain is

$$d(\dot{M}) = d\left(\frac{\partial}{\partial t}(\rho V)\right) = -\rho\mathbf{q} \cdot d\mathbf{S}$$

Fluid Mechanics - The Continuity Equation 2

Integrate! So total rate of mass gain from V is

$$\frac{\partial}{\partial t}\int_V \rho(\mathbf{r})dV = -\int_S \rho\mathbf{q}\cdot d\mathbf{S}.$$

Assuming that the volume of interest is fixed, this is the same as

$$\int_V \frac{\partial\rho}{\partial t}dV = -\int_S \rho\mathbf{q}\cdot d\mathbf{S}\ .$$

Now use Gauss to transform the RHS into a volume integral

$$\int_V \frac{\partial\rho}{\partial t}dV = -\int_V \mathrm{div}\ (\rho\mathbf{q})dV\ .$$

Hence

$$\frac{\partial\rho}{\partial t} = -\mathrm{div}\ (\rho\mathbf{q})$$

Fluid Mechanics - The Continuity Equation 3

To summarize:

The Continuity Equation(s):

$$\mathrm{div}\ (\rho \mathbf{q}) = -\frac{\partial \rho}{\partial t}$$

for time-invariant ρ

$$\mathrm{div}\ (\rho \mathbf{q}) = 0$$

for uniform (space-invariant), time-invariant ρ:

$$\mathrm{div}\ (\mathbf{q}) = 0 \quad . \qquad \mathbf{q}\ \text{solenoidal}$$

3. Thermodynamics: The Heat Conduction Equation

Consider heat flux density $\mathbf{q}(\mathbf{r})$
— heat flow per unit area per unit time.

Assuming
— no mass flow out of control volume
— no source of heat inside control volume ...

$\int_S \mathbf{q} \cdot d\mathbf{S}$ out of control volume by conduction
$=$ decrease of internal energy (constant volume)
$=$ decrease of enthalpy (constant pressure) ...

$$\begin{aligned} \int_S \mathbf{q} \cdot d\mathbf{S} &= -\int_V \rho c \frac{\partial T}{\partial t} dV \\ \Rightarrow \quad \text{div } \mathbf{q} &= -\rho c \frac{\partial T}{\partial t}, \end{aligned}$$

— ρ is constant density of the conducting medium
— c is constant specific heat

Heat conduction ctd ...

To repeat:

$$\int_S \mathbf{q} \cdot d\mathbf{S} = -\int_V \rho c \frac{\partial T}{\partial t} dV \quad \Rightarrow \quad \text{div} \;\; \mathbf{q} = -\rho c \frac{\partial T}{\partial t},$$

To solve for temperature field we need another equation ...

$$\begin{aligned} \mathbf{q} &= -\kappa \,\text{grad}\;\; T \\ -\text{div}\;\; \mathbf{q} &= \kappa \,\text{div}\;\; \text{grad}\;\; T = \kappa \nabla^2 T = \rho c \frac{\partial T}{\partial t} \end{aligned}$$

The heat conduction equation:

$$\nabla^2 T = \frac{\rho c}{\kappa} \frac{\partial T}{\partial t}$$

In steady flow, the h.c.e is Laplace's equation:

$$\nabla^2 T = 0$$

4. Mechanics/Electrostatics - Conservative fields

Recall that a conservative field of force is one for which the work done $\int_A^B \mathbf{F} \cdot d\mathbf{r}$, moving from A to B is **independent of path taken**. Or, equivalently, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$,

Stokes tells us that this is the same as

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0,$$

where S is *any* surface bounded by C.

But if true for *any* C containing A and B, it must be that

curl $\mathbf{F} = \mathbf{0}$ That is **Conservative fields are irrotational**

The only way of satisfying this condition is for $\mathbf{F} = \boldsymbol{\nabla} \ U$

All conservative vector fields have an associated scalar field called the **Potential function** $U(\mathbf{r})$

5. The Inverse Square Law of force

Here's something to prove later ...
All radial vector fields are irrotational.

Radial forces are found in electrostatics and gravitation
— these are are certainly irrotational and conservative.

But in nature these radial forces are also **inverse square laws**.

One reason why this may be so is that inverse square fields turns out to be the only radial fields which are **solenoidal**, i.e. have zero divergence.

How do we show this?

Proof: inverse square radial fields are solenoidal

Let $\mathbf{F} = f(r)\mathbf{r} = rf(r)\hat{\mathbf{r}}$ or $= f(r)(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$,

$$\operatorname{div}\mathbf{a} = \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(a_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi}$$

$$\Rightarrow \operatorname{div}\ \mathbf{F} = \frac{1}{r^2}\frac{\partial(r^3 f(r))}{\partial r} = 3f(r) + r\frac{df}{dr}\ .$$

For div $\mathbf{F} = 0$ we have

$$\Rightarrow r\frac{df}{dr} + 3f = 0 \qquad or \qquad \frac{df}{f} + 3\frac{dr}{r} = 0.$$

Integrate

$$\ln f = -3\ln r + \text{const}$$

$$fr^3 = \text{another const} = k$$

$$\mathbf{F} = \frac{k\mathbf{r}}{r^3}, \quad |\mathbf{F}| = \frac{k}{r^2}.$$

Divergence zero everywhere except origin ...

Zero divergence of the inverse square force field applies everywhere *except* at $\mathbf{r} = \mathbf{0}$. Here its divergence is infinite!

To show this, calculate the outward normal flux out of a sphere of radius R centered on the origin when $\mathbf{F} = F\hat{\mathbf{r}} = (k/r^2)\hat{\mathbf{r}}$. This is

$$\int_S \mathbf{F} \cdot d\mathbf{S} = 4\pi R^2 F = 4\pi R^2 (k/R^2) = 4\pi k = \text{constant} \neq 0$$

Gauss tells us that this flux must be equal to

$$\int_V \text{div } \mathbf{F} dV = \int_0^R \text{div } \mathbf{F}\, 4\pi r^2 dr$$

But for all finite R, $\text{div}\,\mathbf{F} = 0$, so $\text{div}\,\mathbf{F}$ must be infinite at the origin

The flux integral is thus

- zero — for any volume which does not contain the origin
- $4\pi k$ for any volume which does contain it.

6. Gravitational field due to distributed mass: Poisson's Eq

Snag: If one tried this for gravity you would run into the problem that there is no such thing as point mass!
So we deal with distributed mass ...

- Mass in each volume element dV is ρdV.
- Mass inside control volume contributes $4\pi k = -4\pi G \rho dV$ to the flux integral
- Mass outside control volume makes no contribution.

So

$$\int_S \mathbf{F} \cdot d\mathbf{S} = -4\pi G \int_V \rho dV.$$

Transforming the left hand integral by Gauss' Theorem gives

$$\int_V \text{div} \quad \mathbf{F} dV = -4\pi G \int_V \rho dV$$

which, since it is true for any V, implies that

$$\text{div} \quad \mathbf{F} = -4\pi \rho G.$$

Gravitational field, ctd ...

To repeat:

$$\text{div}\quad \mathbf{F} = -4\pi\rho G.$$

But the gravitational field is also conservative & irrotational.
⇒Must have an associated potential function U, and

$$\mathbf{F} = -\text{grad}\ \ U$$

The minus sign is just convention (attractive force)
⇒ the gravitational potential U satisfies

Poisson's Equation

$$\nabla^2 U = 4\pi\rho G\ .$$

7. Gravitational field inside body

Using the integral form of Poisson's equation, it is possible to calculate the gravitational field inside a spherical body whose density is a function of radius only.

We have

$$4\pi R^2 F = 4\pi G \int_0^R 4\pi r^2 \rho dr,$$

where $F = |\mathbf{F}|$

Hence

$$|\mathbf{F}| = \frac{G}{R^2}\int_0^R 4\pi r^2 \rho dr = \frac{MG}{R^2},$$

where M is the total mass inside radius R.

8. Pressure forces in non-uniform flows

Immerse body in flow: it experiences a nett force

$$\mathbf{F}_p = -\int_S p \;\; d\mathbf{S},$$

The integral is taken over the body's entire surface. If pressure p non-uniform, this integral is finite.

Note that the $d\mathbf{F}$ on each surface element is in the direction of the normal to the element.

Now use our extension to Gauss' theorem

$$\mathbf{F}_p = -\int_S p \;\; d\mathbf{S} = -\int_V \text{grad} \;\; p \; dV$$

where V is body's volume.

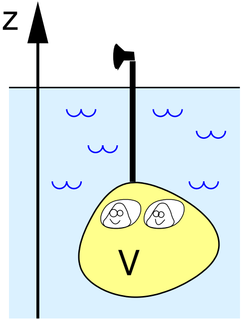
Pressure forces in non-uniform flows

Now at some depth $-z$ (yes, minus, because z points upwards) the Hydrostatic pressure is

$$p = K - \rho g z$$

so that

$$\text{grad } p = -\rho g \hat{\mathbf{k}}$$



and the net pressure force is simply

$$\mathbf{F}_p = g\hat{\mathbf{k}} \int_V \rho dV$$

which, Eureka, is equal to

the weight of fluid displaced.

Summary

This lecture has presented a pot-pourri of applications of vector calculus in analyses of interest to Engineers

We've seen that vector calculus provides a powerful method of describing physical systems in 3 dimensions.