A1 Vector Algebra and Calculus

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Vector Algebra and Calculus

1. Revision of vector algebra, scalar product, vector product
2. Triple products, multiple products, applications to geometry
3. Differentiation of vector functions, applications to mechanics
4. Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates
5. Vector operators — grad, div and curl
6. Vector Identities, curvilinear co-ordinate systems
7. **Gauss' and Stokes' Theorems and extensions**
8. Engineering Applications

Gauss' and Stokes' Theorems

This lecture finally begins to deliver on why we introduced div , grad and curl by introducing ...

- **Gauss' Theorem**
  This enables an integral taken over a volume to be replaced by one taken over the surface bounding that volume, and vice versa.
  Why would we want to do that?
  Computational efficiency and/or numerical accuracy come to mind.

- **Stokes' Theorem**
  This enables an integral taken around a closed curve to be replaced by one taken over *any* surface bounded by that curve.

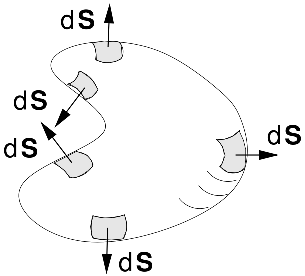
Gauss' Theorem

We want to find the **total outward flux of the vector field $\mathbf{a}(\mathbf{r})$ across the surface S that bounds a volume V**:

$$\int_S \mathbf{a} \cdot d\mathbf{S}$$

$d\mathbf{S}$ is

1. normal to the local surface element
2. must everywhere point out of the volume



Gauss' Theorem tells us that we can do this by considering the total flux generated inside the volume V:

Gauss' Theorem:

$$\int_S \mathbf{a} \cdot d\mathbf{S} = \int_V \mathrm{div}\ \mathbf{a} \ \ dV$$

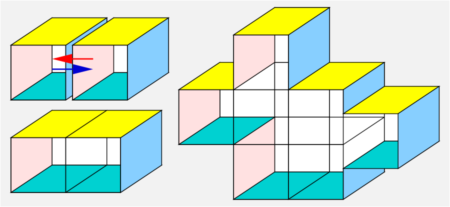
Informal proof

Divergence was *defined* as

$$\text{div } \mathbf{a} \, dV \quad = \quad d(Efflux) \quad = \sum_{\text{surface of } dV} \mathbf{a} \cdot d\mathbf{S} \; .$$

If we sum over the volume elements, this results in a sum over the surface elements.

But if two elemental surface touch, their $d\mathbf{S}$ vectors are in opposing direction and cancel.



Thus the sum over surface elements gives the overall **bounding surface**.

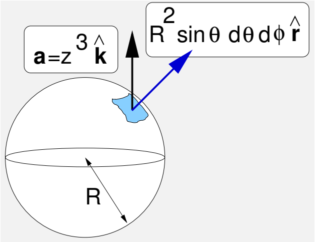
$$\int_V \text{div } \mathbf{a} \;\; dV = \int_{\text{Surface of } V} \mathbf{a} \cdot d\mathbf{S}$$

♣ Example of Gauss' Theorem

Q: Derive $\int_S \mathbf{a} \cdot d\mathbf{S}$ where $\mathbf{a} = z^3\hat{\mathbf{k}}$ and S is the surface of a sphere of radius R centred on the origin: (i) directly; (ii) by applying Gauss' Theorem.

A(i): On the surface of the sphere

$$\begin{aligned}
\mathbf{a} &= R^3 \cos^3\theta \hat{\mathbf{k}} \\
d\mathbf{S} &= h_\theta d\theta h_\phi d\phi (\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}}) = R^2 \sin\theta d\theta d\phi \hat{\mathbf{r}} \\
\mathbf{a} \cdot d\mathbf{S} &= R^3 \cos^3\theta R^2 \sin\theta d\theta d\phi (\hat{\mathbf{r}} \cdot \hat{\mathbf{k}}) \\
\hat{\mathbf{r}} \cdot \hat{\mathbf{k}} &= \cos\theta
\end{aligned}$$



$$\begin{aligned}
\int_S \mathbf{a} \cdot d\mathbf{S} &= \int_{\phi=0}^{2\pi} \int_0^{\pi} R^5 \cos^4\theta \sin\theta d\theta d\phi \\
&= R^5 \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \cos^4\theta \sin\theta d\theta = \frac{2\pi R^5}{5} \left[-\cos^5\theta\right]_0^{\pi} = \frac{4\pi R^5}{5}
\end{aligned}$$

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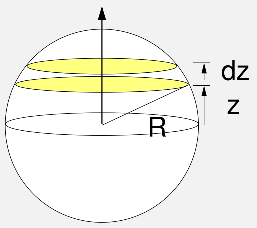
Example /ctd

A(ii): To apply Gauss' Theorem, we need
(i) to work out div **a**

$$\mathbf{a} = z^3\hat{\mathbf{k}}, \quad \Rightarrow \quad \text{div}\,\mathbf{a} = 3z^2$$

(ii) Perform the volume integral. Because div **a** involves just z, we can divide the sphere into discs of constant z and thickness dz. Then



$$dV = \pi(\text{disc radius})^2 dz = \pi(R^2 - z^2)dz$$

So:

$$\int_V \text{div}\ \mathbf{a} dV = 3\pi \int_{-R}^{R} z^2(R^2 - z^2)dz$$

$$= 3\pi \left[\frac{R^2 z^3}{3} - \frac{z^5}{5}\right]_{-R}^{R} = \frac{4\pi R^5}{5}$$

Typical! The surface integral is tedious, but volume integral is "straightforward" ...

An Extension to Gauss' Theorem

Suppose vector field is $\mathbf{a} = U(\mathbf{r})\mathbf{c}$ with $U(\mathbf{r})$ a scalar field & $\mathbf{c}$ a **constant** vector. From Lecture 6 result and noting that $\text{div}\,\mathbf{c} = 0$:

$$\text{div}\;\mathbf{a} = \text{div}\;(U\mathbf{c}) = \text{grad}\,U \cdot \mathbf{c} + U\text{div}\;\mathbf{c} \quad = \text{grad}\,U \cdot \mathbf{c}$$

Gauss' Theorem tells us that

$$\int_S U\mathbf{c} \cdot d\mathbf{S} = \int_V \text{grad}\;\, U \cdot \mathbf{c} dV$$

But taking constant $\mathbf{c}$ outside ...

$$\mathbf{c} \cdot \left(\int_S U d\mathbf{S} \right) = \mathbf{c} \cdot \left(\int_V \text{grad}\;\, U dV \right)$$

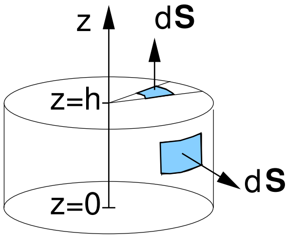
Now $\mathbf{c}$ is arbitrary so result must hold for any vector $\mathbf{c}$. Hence a ...

Gauss-Theorem extension:

$$\int_S U\, d\mathbf{S} = \int_V \text{grad}\, U\;\, dV$$

♣ Example

Q: $U = x^2 + y^2 + z^2$ is a scalar field, and volume V is the cylinder $x^2 + y^2 \leqslant a^2,\ 0 \leqslant z \leqslant h$. Compute the surface integral $\int_S U d\mathbf{S}$ over the surface of the cylinder.



A1) By direct surface integration ...

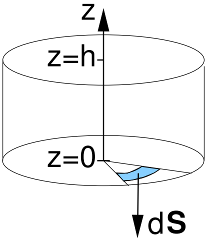
Symmetry gives zero contribution from curved surface, leaving

Top surface:

$$U = (x^2 + y^2 + z^2) = (r^2 + h^2) \quad \text{and} \quad d\mathbf{S} = r dr d\phi \hat{\mathbf{k}}$$

$$\begin{aligned} \Rightarrow \int U d\mathbf{S} &= \int_{r=0}^{a} (h^2 + r^2) r dr \int_{\phi=0}^{2\pi} d\phi \hat{\mathbf{k}} \\ &= \left[\frac{1}{2} h^2 r^2 + \frac{1}{4} r^4 \right]_0^a 2\pi \hat{\mathbf{k}} = \pi \left(h^2 a^2 + \frac{1}{2} a^4 \right) \hat{\mathbf{k}} \end{aligned}$$

♣ Example /ctd



Bottom surface:

$$U = (x^2 + y^2 + z^2) = (x^2 + y^2) = r^2 \quad \& \quad d\mathbf{S} = -rdrd\phi\hat{\mathbf{k}}$$

$$\int Ud\mathbf{S} = -\int_{r=0}^{a} r^3 dr \int_{\phi=0}^{2\pi} d\phi\hat{\mathbf{k}} = -\frac{\pi a^4}{2}\hat{\mathbf{k}}$$

$\Rightarrow$ Total integral is $\quad \pi[h^2a^2 + \frac{1}{2}a^4]\hat{\mathbf{k}} - \frac{1}{2}\pi a^4\hat{\mathbf{k}} = \quad \pi h^2a^2\hat{\mathbf{k}}.$

Example, ctd: the volume integration

To test the RHS of the extension $\int_S U d\mathbf{S} = \int_V \text{grad } U dV$ we have to compute

$$\int_V \text{grad } U dV$$

$$U = x^2 + y^2 + z^2 \quad \Rightarrow \text{grad } U = 2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$$

So the integral is:

$$\begin{aligned}
& 2\int_V (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) r \; dr \; dz \; d\phi \\
= \; & 2\int_{z=0}^{h}\int_{r=0}^{a}\int_{\phi=0}^{2\pi} (r\cos\phi\hat{\mathbf{i}} + r\sin\phi\hat{\mathbf{j}} + z\hat{\mathbf{k}}) r \; dr \; dz \; d\phi \\
= \; & 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 2\int_{z=0}^{h} z dz \int_{r=0}^{a} r \; dr \int_{\phi=0}^{2\pi} d\phi\hat{\mathbf{k}} \quad = \quad \underline{\pi a^2 h^2 \hat{\mathbf{k}}}
\end{aligned}$$

NB: $\hat{\mathbf{i}}$ component is $\propto \int_{\phi=0}^{2\pi} \cos\phi d\phi = 0$ and
the $\hat{\mathbf{j}}$ component is $\propto \int_{\phi=0}^{2\pi} \sin\phi d\phi = 0$

Further extension to Gauss' Theorem

Further "extensions" can be devised ...

For example, apply Gauss' theorem to

$$\mathbf{a}(\mathbf{r}) = \mathbf{b}(\mathbf{r}) \times \mathbf{c}$$

where **c** is a constant vector ...

... and see what happens.

Stokes' Theorem

Stokes' Theorem relates a line integral around a closed path ...

... to a surface integral over what is called a *capping surface* of the path.

Stokes' Theorem:

$$\oint_C \mathbf{a} \cdot d\mathbf{r} = \int_S \text{curl } \mathbf{a} \cdot d\mathbf{S}$$

where S is *any* surface capping the curve C.

Note, RHS is $\int (\text{curl } \mathbf{a}) \cdot d\mathbf{S}$.

Why couldn't it be $\int \text{curl} \, (\mathbf{a} \cdot d\mathbf{S})$?

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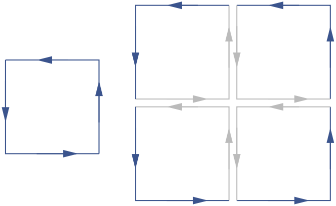
Informal proof

Lecture 5 defined curl as the circulation per unit area, and showed that

$$\sum_{\text{around elemental loop}} \mathbf{a} \cdot d\mathbf{r} = dC = (\nabla \times \mathbf{a}) \cdot d\mathbf{S} \quad .$$

If we add these little loops together, the internal line sections cancel out because the $d\mathbf{r}$'s are in opposite direction but the field $\mathbf{a}$ is not. This gives the larger surface and the larger bounding contour.

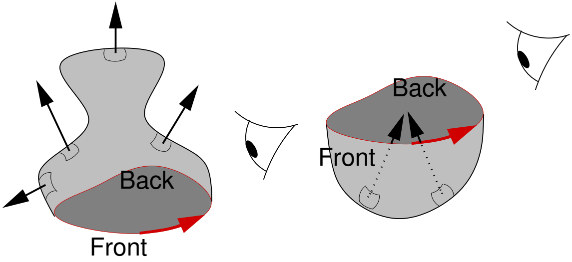


In these diagrams the contour appears planar. In general the contour is any non-intersecting space curve.

Capping Surface

The previous argument says that for a given contour, the capping surface can be **ANY** surface bound by the contour.

Only requirement:
the surface element vectors point in the "general direction" of a r-h screw w.r.t. to the sense of the contour integral.



In practice, (in exam questions?!) the bounding contour is often planar, and the capping surface either flat, or hemispherical, or cylindrical.

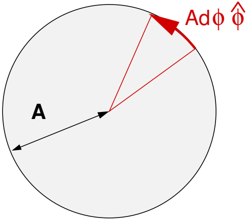
♣ Example of Stokes' Theorem

Q: Field $\mathbf{a} = -y^3\hat{\mathbf{\imath}} + x^3\hat{\mathbf{\jmath}}$ and C is the circle, radius A, centred at $(0,0)$. Derive $\oint_C \mathbf{a}\cdot d\mathbf{r}$ (i) directly and (ii) using Stokes' with a planar surface.

A: Directly

On the circle of radius A

$$\mathbf{a} = A^3(-\sin^3\phi\hat{\mathbf{\imath}} + \cos^3\phi\hat{\mathbf{\jmath}})$$



and

$$d\mathbf{r} = Ad\phi\hat{\boldsymbol{\phi}} = Ad\phi(-\sin\phi\hat{\mathbf{\imath}} + \cos\phi\hat{\mathbf{\jmath}})$$

so that:

$$\oint_C \mathbf{a}\cdot d\mathbf{r} = \int_0^{2\pi} A^4(\sin^4\phi + \cos^4\phi)d\phi = \underline{\underline{\frac{3\pi}{2}A^4}},$$

$$\int_0^{2\pi}\sin^4\phi d\phi = \int_0^{2\pi}\cos^4\phi d\phi = \frac{3\pi}{4}$$

Example /ctd

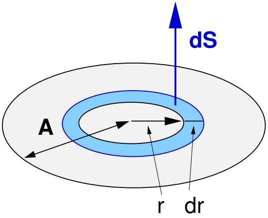
A: (Using Stokes') $\int \text{curl}\,\mathbf{a}\cdot\, d\mathbf{S}$ over planar disc ...

$$\text{curl}\quad \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = 3(x^2+y^2)\hat{\mathbf{k}} = 3r^2\hat{\mathbf{k}}$$

We choose area elements to be circular strips of radius r thickness dr. Then

$$d\mathbf{S} = r dr d\phi \hat{\mathbf{k}}$$

$$\int_S \text{curl}\,\mathbf{a}\cdot d\mathbf{S} = 3\int_{r=0}^{A} r^3 dr \int_{\phi=0}^{2\pi} d\phi = \frac{3\pi}{2}A^4$$



An Extension to Stokes' Theorem

Try similar "extension" with Stokes ...

Again let $\mathbf{a}(\mathbf{r}) = U(\mathbf{r})\mathbf{c}$, where $\mathbf{c}$ is a constant vector. Then

$$\text{curl }\ \mathbf{a} = U\text{curl }\ \mathbf{c} + \text{grad } U \times \mathbf{c}$$

But curl $\mathbf{c}$ is zero. Stokes' Theorem becomes:

$$\oint_C U(\mathbf{c}\cdot d\mathbf{r}) = \int_S \text{grad } U \times \mathbf{c}\cdot d\mathbf{S} = \int_S \mathbf{c}\cdot(d\mathbf{S}\times \text{grad } U)$$

Re-arranging and taking constant $\mathbf{c}$ out ...

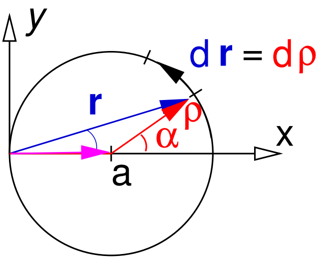
$$\mathbf{c}\cdot\oint_C U d\mathbf{r} = -\mathbf{c}\cdot\int_S \text{grad } U \times d\mathbf{S} \quad .$$

But $\mathbf{c}$ is arbitrary and so

An extension to Stokes': $\oint_C U d\mathbf{r} = -\int_S \text{grad } U \times d\mathbf{S}$

♣ Example of extension to Stokes' Theorem

Q: Derive $\oint_C U d\mathbf{r}$ where $U = x^2 + y^2 + z^2$ and C is the circle $(x-a)^2 + y^2 = a^2, z = 0$, (i) directly and (ii) using Stokes with a planar capping surface.



A(i) Directly: The circle is $\mathbf{r} = a(1+\cos\alpha)\hat{\mathbf{i}} + a\sin\alpha\hat{\mathbf{j}}$, so

$$U = x^2 + y^2 + z^2 = a^2(1+\cos\alpha)^2 + a^2\sin^2\alpha = 2a^2(1+\cos\alpha)$$

$$d\mathbf{r} = a\ d\alpha(-\sin\alpha\hat{\mathbf{i}}\ +\ \cos\alpha\hat{\mathbf{j}})\quad .$$

So,

$$\oint U d\mathbf{r} = 2a^3\int_{\alpha=0}^{2\pi}(1+\cos\alpha)(-\sin\alpha\hat{\mathbf{i}} + \cos\alpha\hat{\mathbf{j}})d\alpha = 2\pi a^3\hat{\mathbf{j}}$$

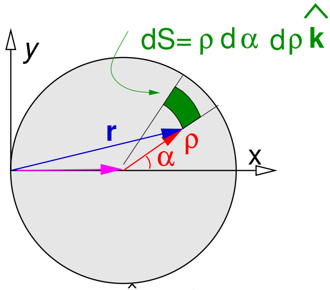
(It is worth checking that a zero $\hat{\mathbf{i}}$ component is indeed what you would expect from symmetry.)

Example /ctd

A(ii) Using Stokes'

Using the *xy*-planar surface

$$\begin{aligned} d\mathbf{S} &= \rho \; d\rho \; d\alpha \; \hat{\mathbf{k}} \\ \text{grad } \; U &= \text{grad } r^2 = 2\mathbf{r} \\ &= 2(a + \rho\cos\alpha)\hat{\mathbf{\imath}} + 2\rho\sin\alpha\hat{\mathbf{\jmath}} \quad , \end{aligned}$$



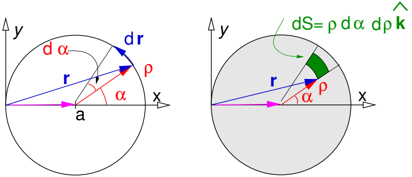
so that

$$\begin{aligned} d\mathbf{S} \times \text{grad } U &= \rho \; d\rho \; d\alpha[2(a+\rho\cos\alpha)(\hat{\mathbf{k}} \times \hat{\mathbf{\imath}}) + 2\rho\sin\alpha(\hat{\mathbf{k}} \times \hat{\mathbf{\jmath}})] \\ &= \rho \; d\rho \; d\alpha[-2\rho\sin\alpha\hat{\mathbf{\imath}} + 2(a+\rho\cos\alpha)\hat{\mathbf{\jmath}}] \end{aligned}$$

and as $\int_0^{2\pi} \sin\alpha d\alpha = 0$ and $\int_0^{2\pi} \cos\alpha d\alpha = 0$

$$\begin{aligned} \int_S d\mathbf{S} \times \text{grad } U &= \int_{\rho=0}^{a} \int_{\alpha=0}^{2\pi} \rho \; d\rho \; d\alpha (2a\hat{\mathbf{\jmath}}) \\ &= 2\pi\frac{a^2}{2}(2a\hat{\mathbf{\jmath}}) = 2\pi a^3 \hat{\mathbf{\jmath}} \end{aligned}$$

Why were ρ, α used in the last eg?



It is simply a coordinate transformation to decouple the coordinates. In the plane the general position is

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}} = (a + \rho\cos\alpha)\hat{\mathbf{i}} + +\rho\sin\alpha\hat{\mathbf{j}}$$

Going round the circumference, both r and θ change, so

$$d\mathbf{r} = (\cos\theta dr - r\sin\theta d\theta)\hat{\mathbf{i}} + (\sin\theta dr + r\cos\theta d\theta)\hat{\mathbf{j}}$$

whereas because $|\boldsymbol{\rho}| = a$ is constant

$$d\boldsymbol{\rho} = (-a\sin\alpha d\alpha)\hat{\mathbf{i}} + (a\cos\alpha d\alpha)\hat{\mathbf{j}}$$

Summary

In this lecture, we have developed

- **Gauss' Theorem**

$$\int_V \text{div } \mathbf{a} \;\; dV = \int_S \mathbf{a} \cdot d\mathbf{S}$$

If you sum up the δ(Effluxes) from each δ(Volume) in an object, you must end up with the overall efflux from the surface.

- **Stokes' Theorem**

$$\oint_C \mathbf{a} \cdot d\mathbf{r} = \int_S \text{curl } \mathbf{a} \cdot d\mathbf{S}$$

which says if you add up the δ(Circulations) per unit area over an open surface, you end up with the Circulation around the rim

- We've seen how to verify and apply the theorems to simple surfaces and volumes using Cartesians, cylindrical and spherical polars.