

A1 Vector Algebra and Calculus

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Vector Algebra and Calculus

- 1 Revision of vector algebra, scalar product, vector product
- 2 Triple products, multiple products, applications to geometry
- 3 Differentiation of vector functions, applications to mechanics
- 4 Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates
- 5 Vector operators — grad, div and curl
- 6 **Vector Identities, curvilinear co-ordinate systems**
- 7 Gauss' and Stokes' Theorems and extensions
- 8 Engineering Applications

Vector Operator Identities & Curvi Coords

In this lecture we look at identities built from vector operators.

These operators behave both as vectors and as differential operators, so that the usual rules of taking the derivative of, say, a product must be observed.

We are laying the groundwork for the use of these identities in later parts of the Engineering course.

We then turn to derive expressions for grad, div and curl in curvilinear coordinates.

After deriving general expressions, we will specialize to the Polar family.



Identity 1: curl grad $U = 0$

$U(x, y, z)$ is a scalar field.

Then

$$\begin{aligned}\nabla \times \nabla U &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial U/\partial x & \partial U/\partial y & \partial U/\partial z \end{vmatrix} \\ &= \hat{\mathbf{i}} \left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y} \right) + \hat{\mathbf{j}} () + \hat{\mathbf{k}} () \\ &= \mathbf{0} .\end{aligned}$$

$\nabla \times \nabla$ can be thought of as a null operator.

Identity 2: $\text{div curl } \mathbf{a} = 0$

For $\mathbf{a}(x, y, z)$ a vector field:

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{a}) &= \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix} \\ &= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial x \partial z} \\ &\quad - \frac{\partial^2 a_z}{\partial y \partial x} + \frac{\partial^2 a_x}{\partial y \partial z} \\ &\quad + \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial z \partial y} \\ &= 0\end{aligned}$$

Identity 3: divergence of $U\mathbf{v}$

$U(\mathbf{r})$ is a scalar field and $\mathbf{v}(\mathbf{r})$ is a vector field.

Eg, $U(\mathbf{r})$ could be fluid density, and $\mathbf{v}(\mathbf{r})$ its instantaneous velocity. Then $U\mathbf{v}$ = mass flux per unit area.

We are interested in the divergence of the product $U\mathbf{v}$.

$$\nabla \cdot (U\mathbf{v}) = U(\nabla \cdot \mathbf{v}) + (\nabla U) \cdot \mathbf{v} = U \operatorname{div} \mathbf{v} + (\operatorname{grad} U) \cdot \mathbf{v}$$

In steps:

$$\begin{aligned} \nabla \cdot (U\mathbf{v}) &= \left(\frac{\partial}{\partial x}(Uv_x) + \frac{\partial}{\partial y}(Uv_y) + \frac{\partial}{\partial z}(Uv_z) \right) \\ &= U \frac{\partial v_x}{\partial x} + U \frac{\partial v_y}{\partial y} + U \frac{\partial v_z}{\partial z} + v_x \frac{\partial U}{\partial x} + v_y \frac{\partial U}{\partial y} + v_z \frac{\partial U}{\partial z} \\ &= U \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \operatorname{grad} U \end{aligned}$$

Identity 3: curl of $U\mathbf{a}$

In a similar way, we can take the curl of the product of a scalar and vector field $U\mathbf{v}$.

The result should be a vector field.

And you're probably happy now to write down

$$\nabla \times (U\mathbf{v}) = U(\nabla \times \mathbf{v}) + (\nabla U) \times \mathbf{v} .$$

Identity 4: $\text{div of } \mathbf{a} \times \mathbf{b}$

But things get trickier to guess when vector or scalar products are involved! Eg, not at all obvious that:

$$\text{div}(\mathbf{a} \times \mathbf{b}) = \text{curl } \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \text{curl } \mathbf{b}$$

Writing $\nabla \cdot (\mathbf{a} \times \mathbf{b})$ indicates that the you could work this like a scalar triple product

$$\begin{aligned} & \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\ &= \frac{\partial}{\partial x} [a_y b_z - a_z b_y] + \frac{\partial}{\partial y} [a_z b_x - a_x b_z] + \frac{\partial}{\partial z} [a_x b_y - a_y b_x] \\ &= \dots \end{aligned}$$

Vector operator identities in HLT

We could carry on inventing vector identities for some time, but ...

Why bother at all, as they are in HLT?

- 1 Since grad , div and curl describe key aspects of vectors fields, they often arise often in practice.

The identities can save you a lot of time and hacking of partial derivatives, as we will see when we consider Maxwell's equation as an example later.

- 2 Secondly, they help to identify other practically important vector operators.

We now look at such an example.

Identity 5: $\text{curl}(\mathbf{a} \times \mathbf{b})$

$$\text{curl}(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x \end{vmatrix}$$

$$\Rightarrow \text{curl}(\mathbf{a} \times \mathbf{b})_x = \frac{\partial}{\partial y}(a_x b_y - a_y b_x) - \frac{\partial}{\partial z}(a_z b_x - a_x b_z)$$

Write as sum of four terms, but add and subtract RED and BLUE terms

$$a_x \left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z} \right) - b_x \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) +$$

$$\left[b_x \frac{\partial}{\partial x} + b_y \frac{\partial}{\partial y} + b_z \frac{\partial}{\partial z} \right] a_x - \left[a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z} \right] b_x$$

Hence, gathering in the y and z components too:

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} + [\mathbf{b} \cdot \nabla]\mathbf{a} - [\mathbf{a} \cdot \nabla]\mathbf{b}$$

$[\mathbf{a} \cdot \nabla]$ can be regarded as new, useful, scalar differential operator.

Definition of the operator $[\mathbf{a} \cdot \nabla]$

This is a *scalar operator* ...

$$[\mathbf{a} \cdot \nabla] \equiv \left[a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z} \right] .$$

Notice that the components of \mathbf{a} don't get touched by the differentiation.

Applied to a scalar field, results in a scalar field

Applied to a vector field results in a vector field

Identity 6: $\text{curl}(\text{curl } \mathbf{a})$ for you to derive

Amuse yourself by deriving the following important identity ...

$$\text{curl}(\text{curl } \mathbf{a}) = \text{grad}(\text{div } \mathbf{a}) - \nabla^2 \mathbf{a}$$

where

$$\nabla^2 \mathbf{a} = \nabla^2 a_x \hat{\mathbf{i}} + \nabla^2 a_y \hat{\mathbf{j}} + \nabla^2 a_z \hat{\mathbf{k}}$$

We are about to use it

♣ Eg using Identity 6: electromagnetic waves

Background: James Clerk Maxwell established (1865) a set of four vector equations, fundamental to working out how electromagnetic waves propagate. The entire telecommunications industry is built on these!

$$\operatorname{div} \mathbf{D} = \rho$$

$$\operatorname{div} \mathbf{B} = 0$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}$$

$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D}$$

In addition, we can assume the following

$$\mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E}$$

$$\mathbf{B} = \mu_r \mu_0 \mathbf{H}$$

$$\mathbf{J} = \sigma \mathbf{E} \text{ grown-up Ohm's law}$$



30 January 1858 "... I have been lecturing on statical electricity to the 2nd year, and next week I shall have half a dozen to study *electrical images* over a cup of tea..."

Example ctd

Q: Show that in a material with no free charge, $\rho = 0$, and with zero conductivity, $\sigma = 0$, the electric field \mathbf{E} must be a solution of the wave equation $\nabla^2 \mathbf{E} = \mu_r \mu_0 \epsilon_r \epsilon_0 (\partial^2 \mathbf{E} / \partial t^2)$.

A:

$$\operatorname{div} \mathbf{D} = \operatorname{div} (\epsilon_r \epsilon_0 \mathbf{E}) = \epsilon_r \epsilon_0 \operatorname{div} \mathbf{E} = \rho = 0; \Rightarrow \operatorname{div} \mathbf{E} = 0$$

$$\operatorname{div} \mathbf{B} = \operatorname{div} (\mu_r \mu_0 \mathbf{H}) = \mu_r \mu_0 \operatorname{div} \mathbf{H} = 0 \quad \Rightarrow \operatorname{div} \mathbf{H} = 0$$

$$\operatorname{curl} \mathbf{E} = -\partial \mathbf{B} / \partial t = -\mu_r \mu_0 (\partial \mathbf{H} / \partial t)$$

$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \partial \mathbf{D} / \partial t = \mathbf{0} + \epsilon_r \epsilon_0 (\partial \mathbf{E} / \partial t)$$

But

$$\operatorname{curl} \operatorname{curl} \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}, \text{ so}$$

$$\operatorname{curl} [-\mu_r \mu_0 (\partial \mathbf{H} / \partial t)] = -\nabla^2 \mathbf{E}$$

$$-\mu_r \mu_0 \frac{\partial}{\partial t} [\operatorname{curl} \mathbf{H}] = -\nabla^2 \mathbf{E}$$

Then finally:

$$-\mu_r \mu_0 \frac{\partial}{\partial t} \left[\epsilon_r \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right] = -\nabla^2 \mathbf{E}$$

$$\Rightarrow \mu_r \mu_0 \epsilon_r \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E}$$

Grad, div, curl and ∇^2 in curvilinear coords

It is possible to obtain general expressions for grad, div and curl in any orthogonal curvilinear co-ordinate system ...

We would guess that we'll need the scale factors h ...

Recall that the unit vector in the direction of increasing u , with v and w being kept constant, is $\hat{\mathbf{u}} = \frac{1}{h_u} \frac{\partial \mathbf{r}}{\partial u}$ where \mathbf{r} is the general position vector,

and $h_u = \left| \frac{\partial \mathbf{r}}{\partial u} \right|$ and similar expressions apply for v and w .

Then

$$d\mathbf{r} = h_u du \hat{\mathbf{u}} + h_v dv \hat{\mathbf{v}} + h_w dw \hat{\mathbf{w}} .$$

Grad in curvilinear coordinates

Using the gradient of a scalar field ϕ ,

$$\nabla\phi \cdot d\mathbf{r} = d\phi \quad \text{and} \quad d\phi = \frac{\partial\phi}{\partial u} du + \frac{\partial\phi}{\partial v} dv + \frac{\partial\phi}{\partial w} dw$$

It follows that

$$\nabla\phi \cdot (h_u \hat{\mathbf{u}} du + h_v \hat{\mathbf{v}} dv + h_w \hat{\mathbf{w}} dw) = \frac{\partial\phi}{\partial u} du + \frac{\partial\phi}{\partial v} dv + \frac{\partial\phi}{\partial w} dw$$

The only way this can be satisfied for independent du , dv , dw is when

Grad ϕ in curvilinear coords:

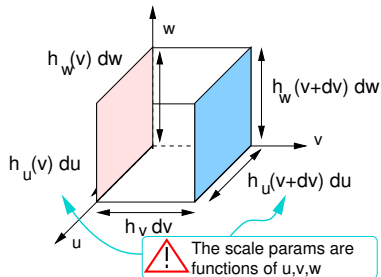
$$\nabla\phi = \frac{1}{h_u} \frac{\partial\phi}{\partial u} \hat{\mathbf{u}} + \frac{1}{h_v} \frac{\partial\phi}{\partial v} \hat{\mathbf{v}} + \frac{1}{h_w} \frac{\partial\phi}{\partial w} \hat{\mathbf{w}}$$

Divergence in curvilinear coordinates

If the curvilinear coordinates are orthogonal then δVolume is a cuboid (to 1st order in small things) and

$$dV = h_u h_v h_w du dv dw .$$

BUT it is not quite a cuboid: the area of two opposite faces will differ as the **scale parameters are functions of u, v, w** .



So the net efflux from the two faces in the \hat{v} dirn is

$$\left[a_v + \frac{\partial a_v}{\partial v} dv \right] \left[h_u + \frac{\partial h_u}{\partial v} dv \right] \left[h_w + \frac{\partial h_w}{\partial v} dv \right] dudw - a_v h_u h_w dudw$$

$$\approx \frac{\partial (a_v h_u h_w)}{\partial v} dudvdw$$

Divergence in curvilinear coordinates /ctd

Repeat: the net efflux from the two faces in the \mathbf{v} dirn is

$$\left[a_v + \frac{\partial a_v}{\partial v} dv \right] \left[h_u + \frac{\partial h_u}{\partial v} dv \right] \left[h_w + \frac{\partial h_w}{\partial v} dv \right] dudw - a_v h_u h_w dudw \\ \approx \frac{\partial(a_v h_u h_w)}{\partial v} dudvdw$$

Now div is net efflux per unit volume, so sum up other faces:

$$\operatorname{div} \mathbf{a} dV = \left(\frac{\partial(a_u h_v h_w)}{\partial u} + \frac{\partial(a_v h_u h_w)}{\partial v} + \frac{\partial(a_w h_u h_v)}{\partial w} \right) dudvdw$$

Then divide by $dV = h_u h_v h_w dudvdw \dots$

div in curvilinear coords is:

$$\operatorname{div} \mathbf{a} = \frac{1}{h_u h_v h_w} \left(\frac{\partial(a_u h_v h_w)}{\partial u} + \frac{\partial(a_v h_u h_w)}{\partial v} + \frac{\partial(a_w h_u h_v)}{\partial w} \right)$$

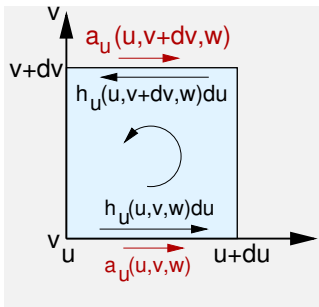
Curl in curvilinear coordinates

For an orthogonal curvi coord system
 $dS = h_u h_v du dv$.

But the opposite sides are not of same length!

The lengths are

$$h_u(u, v, w) du, \text{ and } h_u(u, v+dv, w) du.$$



Summing this pair contributes to circulation (in $\hat{\mathbf{w}}$ dirn)

$$a_u(u, v, w) h_u(u, v, w) du - a_u(u, v+dv, w) h_u(u, v+dv, w) du = -\frac{\partial(h_u a_u)}{\partial v} dv du$$

Add in the other pair to find circulation per unit area

$$\frac{dC}{h_u h_v du dv} = \frac{1}{h_u h_v} \left(-\frac{\partial(h_u a_u)}{\partial v} + \frac{\partial(h_v a_v)}{\partial u} \right)$$

Curl in curvilinear coordinates, ctd

To repeat, the part related to $\hat{\mathbf{w}}$ is:

$$\frac{dC}{h_u h_v du dv} = \frac{1}{h_u h_v} \left(-\frac{\partial(h_u a_u)}{\partial v} + \frac{\partial(h_v a_v)}{\partial u} \right)$$

Adding in the other two components gives:

$$\begin{aligned} \text{curl } \mathbf{a}(u, v, w) = & \frac{1}{h_v h_w} \left(\frac{\partial(h_w a_w)}{\partial v} - \frac{\partial(h_v a_v)}{\partial w} \right) \hat{\mathbf{u}} + \\ & \frac{1}{h_w h_u} \left(\frac{\partial(h_u a_u)}{\partial w} - \frac{\partial(h_w a_w)}{\partial u} \right) \hat{\mathbf{v}} + \\ & \frac{1}{h_u h_v} \left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v} \right) \hat{\mathbf{w}} \end{aligned}$$

You should show that can be written more compactly as:

Curl in curvi coords is:

$$\text{curl } \mathbf{a}(u, v, w) = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \hat{\mathbf{u}} & h_v \hat{\mathbf{v}} & h_w \hat{\mathbf{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u a_u & h_v a_v & h_w a_w \end{vmatrix}$$

The Laplacian in curvilinear coordinates

Substitute the components of $\text{grad } \phi$ into the expression for $\text{div } \mathbf{a}$...

Much grinding gives the following expression for the Laplacian in general orthogonal co-ordinates:

Laplacian in curvilinear coords is:

$$\nabla^2 U = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} \left(\frac{h_v h_w}{h_u} \frac{\partial U}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_w h_u}{h_v} \frac{\partial U}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_u h_v}{h_w} \frac{\partial U}{\partial w} \right) \right].$$

Grad, etc, the 3D polar coordinate

There is no need slavishly to memorize the above derivations or their results.

More important is to realize why the expressions look suddenly more complicated in curvilinear coordinates

We are now going to specialize our expressions for the polar family

As they are 3D entities, we need consider only cylindrical and spherical polars.

Grad, etc, in cylindrical polars

Recall that $\mathbf{r} = r \cos \theta \hat{\mathbf{i}} + r \sin \theta \hat{\mathbf{j}} + z \hat{\mathbf{k}}$, and that $h_u = |\partial \mathbf{r} / \partial u|$, and so

$$h_r = \sqrt{(\cos^2 \theta + \sin^2 \theta)} = 1,$$

$$h_\theta = \sqrt{(r^2 \sin^2 \theta + r^2 \cos^2 \theta)} = r,$$

$$h_z = 1$$

Hence, using these and $U(r, \theta, z)$ and $\mathbf{a} = a_r \hat{\mathbf{r}} + a_\theta \hat{\boldsymbol{\theta}} + a_\phi \hat{\boldsymbol{\phi}}$

$$\text{grad } U = \frac{\partial U}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial U}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{\partial U}{\partial z} \hat{\mathbf{k}}$$

$$\text{div } \mathbf{a} = \frac{1}{r} \left(\frac{\partial (r a_r)}{\partial r} + \frac{\partial a_\theta}{\partial \theta} \right) + \frac{\partial a_z}{\partial z}$$

$$\text{curl } \mathbf{a} = \left(\frac{1}{r} \frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z} \right) \hat{\mathbf{r}} + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r} \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left(\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta} \right) \hat{\mathbf{k}}$$

The derivation of the expression for $\nabla^2 U$ in cylindrical polar co-ordinates is set as a tutorial exercise.



Grad, etc, in spherical polars

We recall that $\mathbf{r} = r \sin \theta \cos \phi \hat{\mathbf{i}} + r \sin \theta \sin \phi \hat{\mathbf{j}} + r \cos \theta \hat{\mathbf{k}}$ so that

$$h_r = \sqrt{(\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta)} = 1$$

$$h_\theta = \sqrt{(r^2 \cos^2 \theta (\cos^2 \phi + \sin^2 \phi) + r^2 \sin^2 \theta)} = r$$

$$h_\phi = \sqrt{(r^2 \sin^2 \theta (\sin^2 \phi + \cos^2 \phi))} = r \sin \theta$$

$$\text{grad } U = \frac{\partial U}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial U}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} \hat{\boldsymbol{\phi}}$$

$$\text{div } \mathbf{a} = \frac{1}{r^2} \frac{\partial (r^2 a_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (a_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial a_\phi}{\partial \phi}$$

$$\begin{aligned} \text{curl } \mathbf{a} &= \frac{\hat{\mathbf{r}}}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (a_\phi \sin \theta) - \frac{\partial}{\partial \phi} (a_\theta) \right) \\ &+ \frac{\hat{\boldsymbol{\theta}}}{r \sin \theta} \left(\frac{\partial}{\partial \phi} (a_r) - \frac{\partial}{\partial r} (a_\phi r \sin \theta) \right) + \frac{\hat{\boldsymbol{\phi}}}{r} \left(\frac{\partial}{\partial r} (a_\theta r) - \frac{\partial}{\partial \theta} (a_r) \right) \end{aligned}$$

♣ Examples

Q:

Find $\text{curl } \mathbf{a}$ in

- Cartesians, and
- Spherical polars

when $\mathbf{a} = x(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$.

A(i):

In Cartesians, using the pseudo determinant gives

$$\text{curl } \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & xz \end{vmatrix} = -z\hat{\mathbf{j}} + y\hat{\mathbf{k}}$$

♣ Example / ctd

A(ii):

We were told $\mathbf{a} = x(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$.

In spherical polars $x = r \sin \theta \cos \phi$ and $(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = \mathbf{r}$

Hence $\mathbf{a} = r \sin \theta \cos \phi \mathbf{r} = r^2 \sin \theta \cos \phi \hat{\mathbf{r}}$

or in component form: $a_r = r^2 \sin \theta \cos \phi$; $a_\theta = 0$; $a_\phi = 0$.

Expression for curl (earlier, and HLT):

$$\begin{aligned} \text{curl } \mathbf{a} &= \frac{\hat{\mathbf{r}}}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (a_\phi \sin \theta) - \frac{\partial}{\partial \phi} (a_\theta) \right) + \frac{\hat{\boldsymbol{\theta}}}{r \sin \theta} \left(\frac{\partial}{\partial \phi} (a_r) - \frac{\partial}{\partial r} (a_\phi r \sin \theta) \right) \\ &+ \frac{\hat{\boldsymbol{\phi}}}{r} \left(\frac{\partial}{\partial r} (a_\theta r) - \frac{\partial}{\partial \theta} (a_r) \right) \end{aligned}$$

$$\begin{aligned} \Rightarrow \text{curl } \mathbf{a} &= \frac{\hat{\boldsymbol{\theta}}}{r \sin \theta} \left(\frac{\partial}{\partial \phi} (r^2 \sin \theta \cos \phi) \right) + \frac{\hat{\boldsymbol{\phi}}}{r} \left(-\frac{\partial}{\partial \theta} (r^2 \sin \theta \cos \phi) \right) \\ &= \frac{\hat{\boldsymbol{\theta}}}{r \sin \theta} (-r^2 \sin \theta \sin \phi) + \frac{\hat{\boldsymbol{\phi}}}{r} (-r^2 \cos \theta \cos \phi) \\ &= \hat{\boldsymbol{\theta}} (-r \sin \phi) + \hat{\boldsymbol{\phi}} (-r \cos \theta \cos \phi) \end{aligned}$$

Check: These two results should be the same!

To check we need $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$ in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$...

If we were doing this for a perfectly general curvi coord system we would resort to writing the position vector

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Then using

$$\hat{\mathbf{r}} = \frac{1}{h_r} \frac{\partial \mathbf{r}}{\partial r}; \quad \hat{\theta} = \frac{1}{h_\theta} \frac{\partial \mathbf{r}}{\partial \theta}; \quad \hat{\phi} = \frac{1}{h_\phi} \frac{\partial \mathbf{r}}{\partial \phi}$$

Then, doing the first of these, and using $h_r = 1$

$$\begin{aligned} \hat{\mathbf{r}} &= \frac{\partial}{\partial r} \left(r \sin \theta \cos \phi \hat{\mathbf{i}} + r \sin \theta \sin \phi \hat{\mathbf{j}} + r \cos \theta \hat{\mathbf{k}} \right) \\ &= \left(\sin \theta \cos \phi \hat{\mathbf{i}} + \sin \theta \sin \phi \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}} \right) \end{aligned}$$

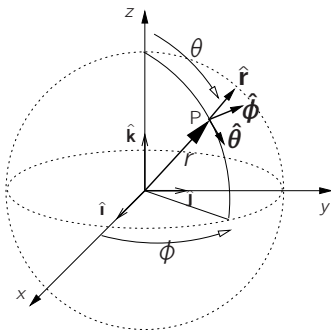
Check: These two results should be the same!

As these are spherical polars, they are easy enough to write down!

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = [R] \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Don't be surprised to see a rotation matrix $[R]$!

We are rotating one right-handed orthogonal coord system into another.



Check /ctd

Now write down our earlier result, and slot in the transformation ...

$$\begin{aligned}
 \text{curl } \mathbf{a} &= \hat{\boldsymbol{\theta}}(-r \sin \phi) + \hat{\boldsymbol{\phi}}(-r \cos \theta \cos \phi) = -r[0, \sin \phi, \cos \theta \cos \phi] \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} \\
 &= -r[0, \sin \phi, \cos \theta \cos \phi] \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} \\
 &= \text{Bish Bash Bosh} \\
 &= -r \cos \theta \hat{\mathbf{j}} + r \sin \theta \sin \phi \hat{\mathbf{k}} \\
 &= -z \hat{\mathbf{j}} + y \hat{\mathbf{k}}
 \end{aligned}$$

This is exactly what we got before!

Rather a lot of work to check, but worth doing.

Summary

Take home messages ...

The key thing when combining operators is to remember that each partial derivative operates on everything to its right.

The identities (eg in HLT) are not mysterious. They merely provide useful short cuts.

There is no need slavishly to learn the expressions for grad, div and curl in curvi coords.

They are in HLT, but

- you need to know how they originate.
- you need to be able to hack them out when asked.

Ditto with the specializations to polars.

Just as physical vectors are independent of their coordinate systems, so are differential operators.