A1 Vector Algebra and Calculus

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Vector Algebra and Calculus

1. Revision of vector algebra, scalar product, vector product
2. Triple products, multiple products, applications to geometry
3. Differentiation of vector functions, applications to mechanics
4. **Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates**
5. Vector operators — grad, div and curl
6. Vector Identities, curvilinear co-ordinate systems
7. Gauss' and Stokes' Theorems and extensions
8. Engineering Applications

We started off

- being concerned with individual vectors **a**, **b**, **c**, and so on.

We went on

- to consider how single vectors vary over time or over some other parameter such as arc length

In rest of the course, we will be concerned with

- scalars and vectors which are defined over regions in space

In this lecture we introduce

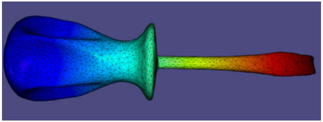
- line, surface and volume integrals
- definition in *curvilinear coordinates*

Scalar and vector fields

If a scalar function $u(\mathbf{r})$ is defined at each $\mathbf{r}$ in some region

- u is a **scalar field** in that region.

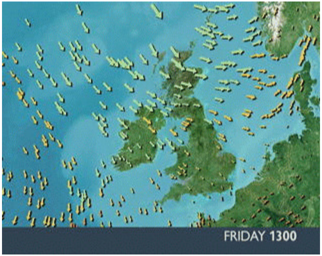
Examples: temperature, pressure, altitude, CO_2 concentration



Similarly, if a vector function $\mathbf{v}(\mathbf{r})$ is defined at each point, then

- $\mathbf{v}$ is a **vector field** in that region.

Examples: wind velocity, magnetic field, traffic flows, optical flow, electric fields



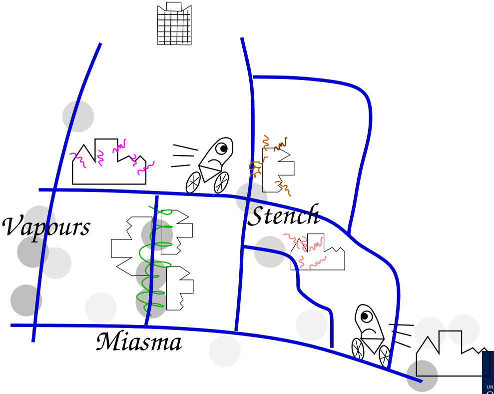
In **field theory** the aim is to derive statements about **bulk properties** of scalar and vector fields

Line integrals through fields

Line integrals are concerned with measuring

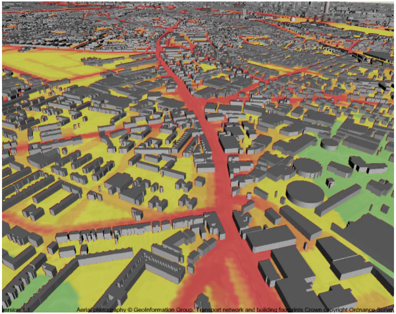
- the integrated interaction with a field as you move through it on some defined path.

Eg, given a map showing the pollution density field in Oxford, how much gunk would you breath in when cycling from college to the Department on different routes?



Not entirely frivolous ...

NO_2 in area of SE London



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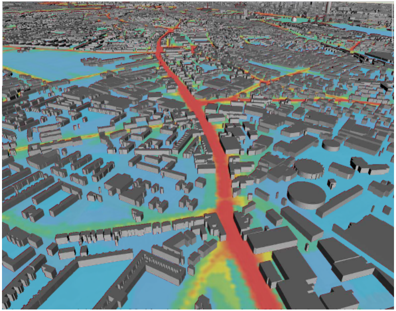

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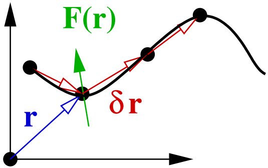
Vector line integrals

1) Chop path L into *vector* segments $\delta\mathbf{r}_i$.

2) Multiply each segment by the field value at that point in space.

3) Sum products.



Three types:

1: Integrand $U(\mathbf{r})$ is a scalar field. Integral is a vector.

$$\mathbf{I} = \int_L U(\mathbf{r}) d\mathbf{r}$$

2: Integrand $\mathbf{a}(\mathbf{r})$ is a vector field dotted with $d\mathbf{r}$. Integral is a scalar:

$$I = \int_L \mathbf{a}(\mathbf{r}) \cdot d\mathbf{r}$$

3: Integrand $\mathbf{a}(\mathbf{r})$ is a vector field crossed with $d\mathbf{r}$. Integral is vector.

$$\mathbf{I} = \int_L \mathbf{a}(\mathbf{r}) \times d\mathbf{r}$$

♣ Examples

Total work done by force $\mathbf{F}$ as it moves point from A to B along path C. Infinitesimal work done is $dW = \mathbf{F} \cdot d\mathbf{r}$, hence total work is

$$W_C = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Ampère's law relating magnetic intensity $\mathbf{H}$ to linked current can be written as

$$I = \oint_C \mathbf{H} \cdot d\mathbf{r}$$

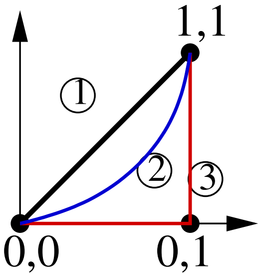
Force on an element of wire carrying current I when placed in a magnetic flux density $\mathbf{B}$ is $d\mathbf{F} = I d\mathbf{r} \times \mathbf{B}$.
So total force on loop of wire C :

$$\mathbf{F} = I \oint_C d\mathbf{r} \times \mathbf{B}$$

Note: expressions above are beautifully compact in vector notation, and are all independent of coordinate system

♣ Examples

Q: A force $\mathbf{F} = x^2 y\hat{\imath} + xy^2\hat{\jmath}$ acts on a body at it moves between $(0,0)$ and $(1,1)$. Find work done when the path is:



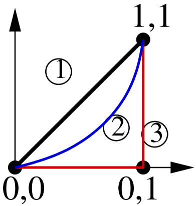
1. along the line $y = x$.
2. along the curve $y = x^n$.
3. along the x axis to the point $(1,0)$ and then along the line $x = 1$

A:
In planar Cartesians $\quad \mathbf{r} = \hat{\imath}x + \hat{\jmath}y \quad \Rightarrow d\mathbf{r} = \hat{\imath}dx + \hat{\jmath}dy$
Then the work done is

$$\int_L \mathbf{F} \cdot d\mathbf{r} = \int_L (x^2 y\hat{\imath} + xy^2\hat{\jmath}) \cdot (\hat{\imath}dx + \hat{\jmath}dy) = \int_L (x^2 y dx + xy^2 dy) \ .$$

Example Path 1

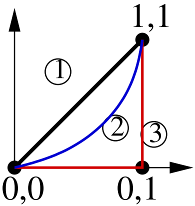
PATH 1: For the path $y = x$ we find that $dy = dx$. So it is easiest to convert all y references to x.



$$\int_{(0,0)}^{(1,1)} (x^2 y dx + x y^2 dy) = \int_{x=0}^{x=1} (x^2 x dx + x x^2 dx)$$
$$= \int_{x=0}^{x=1} 2x^3 dx$$
$$= \left[x^4/2\right|_{x=0}^{x=1} = 1/2 \; .$$

NB! Although x, y involved these are NOT double integrals. Why not?

Example Path 2



PATH 2: For path $y = x^n$

$$dy = nx^{n-1}dx$$

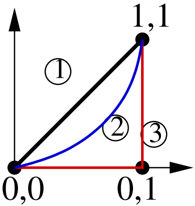
Again convert y references to x.

$$\int_{(0,0)}^{(1,1)} (x^2 y dx + xy^2 dy) = \int_{x=0}^{x=1} (x^{n+2} dx + nx^{n-1}.x.x^{2n} dx)$$

$$= \int_{x=0}^{x=1} (x^{n+2} dx + nx^{3n} dx)$$

$$= \frac{1}{n+3} + \frac{n}{3n+1}$$

Example Path 3



PATH 3: not smooth, so **break into two**. Along the first section, $y = 0$ and $dy = 0$, along second section $x = 1$ and $dx = 0$:

$$\begin{aligned}
\int_A^B (x^2 y dx + xy^2 dy) &= \int_{x=0}^{x=1} (x^2 0 dx) + \int_{y=0}^{y=1} 1.y^2 dy \\
&= 0 + \left[y^3/3\right|_{y=0}^{y=1} \\
&= 1/3 \; .
\end{aligned}$$

Line integral depends on path taken

♣ Example 2

Q2: Repeat path (2), but now using the Force $\mathbf{F} = xy^2\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}}$.
A2:

$$\mathbf{F} \cdot (\hat{\mathbf{i}}dx + \hat{\mathbf{j}}dy) = xy^2\ dx \quad + \quad x^2y\ dy.$$

For the path $y = x^n$ we find that $dy = nx^{n-1}dx$, so

$$\begin{aligned}
\int_{(0,0)}^{(1,1)} (xy^2dx + x^2ydy) \quad &= \quad \int_{x=0}^{x=1} (x^{2n+1}dx + nx^{n-1}x^2x^ndx) \\
&= \quad \int_{x=0}^{x=1} (x^{2n+1}dx + nx^{2n+1}dx) \\
&= \quad \frac{1}{2n+2} + \frac{n}{2n+2} \quad = \quad \frac{1}{2}
\end{aligned}$$

This is independent of n, so

This line is independent of path!

Can we understand why?

Line integrals in Conservative fields

Write

$$g(x, y) = x^2y^2/2$$

Then the perfect differential is

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy \quad = \quad y^2xdx + x^2ydy$$

So our line integral

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_A^B (y^2xdx + yx^2dy) = \int_A^B dg = g_B - g_A$$

It depends solely on the value of g at the start and end points, and not at all on the path

A vector field which gives rise to line integrals which are independent of paths is called **a conservative field**

Some questions about conservative fields

One sort of line integral performs the integration around a complete loop. It is denoted $\oint$

1. If $\mathbf{E}$ is a conservative field, what is the value of $\oint \mathbf{E} \cdot d\mathbf{r}$?
2. If $\mathbf{E}_1$ and $\mathbf{E}_2$ is conservative, is $\mathbf{E}_1 + \mathbf{E}_2$ conservative?
3. Later we show that the electric field around a point charge q

$$\mathbf{E} = Kq\frac{\hat{\mathbf{r}}}{r^2} \qquad K = 1/4\pi\epsilon_r\epsilon_0$$

   is conservative. Are all electric fields conservative?

4. If $\mathbf{E}$ is the electric field, the potential function is

$$\phi = -\int \mathbf{E} \cdot d\mathbf{r} \; .$$

   So are all electric fields conservative?

Line integrals & parametrized curves

♣ Example 1:
Q: Evaluate $\int \mathbf{F} \cdot d\mathbf{r}$ when $\mathbf{F} = z\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$ from $(0,0,0)$ to $(1,1,1)$ along the space curve $x = p$, $y = p^2$, $z = p^3$.
A:

$$
\begin{aligned}
\mathbf{F} &= p^3\hat{\mathbf{i}} + p^4\hat{\mathbf{j}} + p^3\hat{\mathbf{k}} \\
d\mathbf{r} &= dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}} \\
&= dp\hat{\mathbf{i}} + 2p\,dp\hat{\mathbf{j}} + 3p^2\,dp\hat{\mathbf{k}} \\
\int \mathbf{F} \cdot d\mathbf{r} &= \int_{p=0}^{p=1} (p^3\,dp + 2p^5\,dp + 3p^5\,dp) \\
&= \left[(1/4)p^4 + (5/6)p^6\right|_{p=0}^{p=1} \\
&= (26/24)\ .
\end{aligned}
$$

Suppose the integral was from $(0,0,0)$ to $(-2,4,-8)$ ... $\int_{p=?}^{p=?}$

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Line integrals & parametrized curves /ctd

Above, $\int \mathbf{F} \cdot d\mathbf{r}$ boiled down to working out some straightforward $\int F(p)dp$. So, while the following don't appear to involve vectors, they could be the last stage in a vector integral ...

♣ 2:

Consider

$$I = \int_L F(x, y, z) ds \quad ,$$

where the path L is the curve defined as $x = x(p)$, $y = y(p)$, $z = z(p)$. First, convert the function to $F(p)$, writing

$$I = \int_{p_{\textbf{start}}}^{p_{\textbf{end}}} F(p) \frac{ds}{dp}\ dp$$

where (from Lec 3) $\displaystyle \frac{ds}{dp} = \left[\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2 \right]^{1/2}$.

Then do the (now straightforward) integral w.r.t. p.

Line integrals & parametrized curves /ctd

$$I = \int_L F(x, y, z) ds$$

♣ **3:** Suppose parameter is arc-length s and the path L is $x = x(s)$, $y = y(s)$, $z = z(s)$. Convert the function to $F(s)$, writing

$$I = \int_{s_{\textbf{start}}}^{s_{\textbf{end}}} F(s)\ ds$$

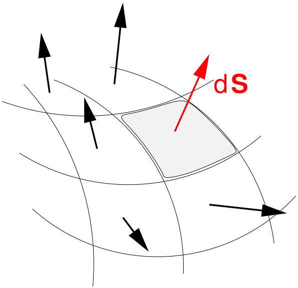
♣ **4:** If p is x — so $y = y(x)$ and $z = z(x)$ (or similar for $p = y$ or $p = z$)

$$I = \int_{x_{\textbf{start}}}^{x_{\textbf{end}}} F(x) \left[1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2 \right]^{1/2} dx \quad .$$

Surface integrals

Surface S is divided into infinitesimal vector elements of area $d\mathbf{S}$:

- the dirn of the vector $d\mathbf{S}$ is the surface normal
- its magnitude represents the area of the element.



Again there are three possibilities:

1: $\int_S U d\mathbf{S}$ — scalar field U; vector integral.

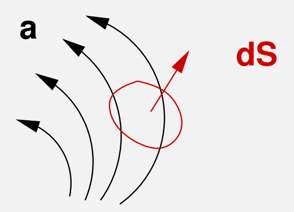
2: $\int_S \mathbf{a} \cdot d\mathbf{S}$ — vector field $\mathbf{a}$; scalar integral.

3: $\int_S \mathbf{a} \times d\mathbf{S}$ — vector field $\mathbf{a}$; vector integral.

Physical example of surface integral

- Physical examples of surface integrals often involve the idea of **flux** of a vector field through a surface

$$\int_S \mathbf{a} \cdot d\mathbf{S}$$



- Mass of fluid crossing a surface element $d\mathbf{S}$ at $\mathbf{r}$ in time dt is

$$dM = \rho\, \mathbf{v} \cdot d\mathbf{S} dt$$

Total rate of gain of mass can be expressed as a surface integral:

$$\frac{dM}{dt} = \int_S \rho(\mathbf{r})\mathbf{v}(\mathbf{r}) \cdot d\mathbf{S}$$

Note that expression is free of any coordinate system

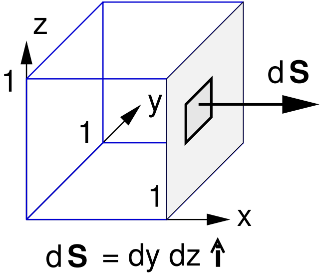
♣ Example

Q: Evaluate $\int \mathbf{F} \cdot d\mathbf{S}$ over the $x = 1$ side of the cube shown in the figure when $\mathbf{F} = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}$.
A: $d\mathbf{S}$ is perp to the surface. Often, the surface will enclose a volume: the surface direction is everywhere out of the volume

For the $x = 1$ face of the cube,

$$d\mathbf{S} = dydz\hat{\mathbf{i}}$$

$$\begin{aligned}
\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}) \cdot dydz\hat{\mathbf{i}} \\
&= \int_{y=0}^{y=1} \int_{z=0}^{z=1} ydydz \\
&= \frac{1}{2}\, y^2\Big|_0^1 \, z\Big|_0^1 \\
&= 1/2 \ .
\end{aligned}$$



Volume integrals

The definition of the volume integral is again taken as the limit of a sum of products as the size of the volume element tends to zero.

One obvious difference though is that the element of volume is a scalar.

The possibilities are:

1: $\int_V U(\mathbf{r})dV$ — scalar field; scalar integral (1P1 stuff!)

2: $\int_V \mathbf{a}(\mathbf{r})dV$ — vector field; vector integral. In this case one can treat each component separately.

$$
\begin{aligned}
\int_V \mathbf{a} dV &= \int_V a_1(x,y,z)\hat{\mathbf{i}}dV + \int_V a_2(x,y,z)\hat{\mathbf{j}}dV + \int_V a_3(x,y,z)\hat{\mathbf{k}}dV \\
&= \hat{\mathbf{i}}\int_V a_1(x,y,z)dV + \hat{\mathbf{j}}\int_V a_2(x,y,z)dV + \hat{\mathbf{k}}\int_V a_3(x,y,z)dV
\end{aligned}
$$

So, $3\times$ 1P1 stuff.

Changing variables: curvilinear coordinates

Before dealing with further examples of line, surface and volume integrals it is important to understand how to convert an integral from one set of coordinates into another

You saw how to do this for scalar volume integrals in 1P1 (and we've seen that volume integrals can always be handled as scalars)

— but we need to understand where Jacobians came from, and how we can apply the mechanism more generally.

You will find the general problem slightly heavy going

— the better news is that later we specialize to the standard set of polar coordinate systems

Changing variables: curvilinear coordinates

The line integral in Cartesian coordinates uses

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad \text{and} \qquad d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$$

You can be sure that length scales are properly handled because

$$|d\mathbf{r}| = ds = \sqrt{dx^2 + dy^2 + dz^2}.$$

But often symmetry screams at you to change coordinate system:

- likely to be plane, cylindrical, or spherical polars,
- but can be something more general like "u, v, w" — a **curvilinear coordinate system**

Now the bad news: Length scales are screwed up

$$\begin{aligned} \mathbf{r} &\neq u\hat{\mathbf{u}} + v\hat{\mathbf{v}} + w\hat{\mathbf{w}} \\ d\mathbf{r} &\neq du\hat{\mathbf{u}} + dv\hat{\mathbf{v}} + dw\hat{\mathbf{w}} \\ |d\mathbf{r}| = ds &\neq \sqrt{du^2 + dv^2 + dw^2} . \end{aligned}$$

Finding the length scales

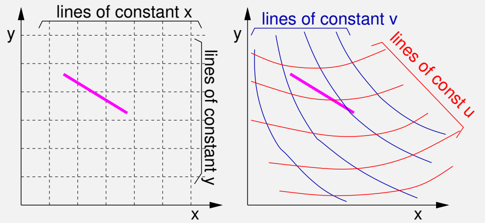
Consider the transform to u, v where $x = x(u,v)$ and $y = y(u,v)$

We write
$\mathbf{r} = x(u,v)\hat{\mathbf{i}} + y(u,v)\hat{\mathbf{j}}$

And because

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$



we can write

$$\begin{aligned} d\mathbf{r} &= \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)\hat{\mathbf{i}} + \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)\hat{\mathbf{j}} \\ &= \left(\frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}}\right)du + \left(\frac{\partial x}{\partial v}\hat{\mathbf{i}} + \frac{\partial y}{\partial v}\hat{\mathbf{j}}\right)dv \\ &= h_u\,\hat{\mathbf{u}}\,du + h_v\,\hat{\mathbf{v}}\,dv \end{aligned}$$

h_u and h_v are called **metric coefficients**

Metric coefficients, ctd

To repeat, the metric coefficients appear in

$$\begin{aligned} d\mathbf{r} &= \left(\frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}}\right) du + \left(\frac{\partial x}{\partial v}\hat{\mathbf{i}} + \frac{\partial y}{\partial v}\hat{\mathbf{j}}\right) dv \\ &= h_u \; \hat{\mathbf{u}} \; du \quad + \quad h_v \; \hat{\mathbf{v}} \; dv \end{aligned}$$

$h_{u,v}$ are the factors that turn the du, dv, or whatever, into proper lengths. But we can also write

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv \quad \Rightarrow h_u \hat{\mathbf{u}} = \frac{\partial \mathbf{r}}{\partial u} \;\;\&\;\; h_v \hat{\mathbf{v}} = \frac{\partial \mathbf{r}}{\partial v}$$

As $\hat{\mathbf{u}}$ has to be a unit vector, we find that

$$h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right| = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2\right]^{1/2}$$

and similarly for $\hat{\mathbf{v}}$

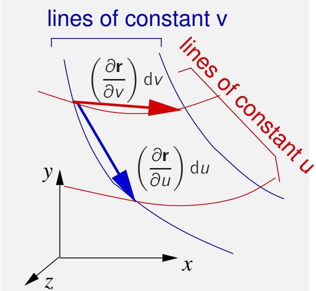
We can tie this in with tangents

If we write the position vector as

$$\mathbf{r} = x(u,v)\hat{\mathbf{\imath}} + y(u,v)\hat{\mathbf{\jmath}}$$

we find the tangent to a v-constant curve as

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\hat{\mathbf{\imath}} + \frac{\partial y}{\partial u}\hat{\mathbf{\jmath}}$$



☛ This is like $d\mathbf{r}/dp$ but is partial because there are two parameters and v is being held constant!

But u is not arclength, so $\partial\mathbf{r}/\partial u$ will not be a unit tangent, rather

$$\frac{\partial \mathbf{r}}{\partial u} = h_u \hat{\mathbf{u}}\ , \quad \text{so} \quad h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right| \qquad \& \quad \hat{\mathbf{u}} = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u}$$

and similarly for $\hat{\mathbf{v}}$. Exactly what we derived before!

To summarize ...

These ideas extend to n-vectors without need for further proof.

Summary

$$\mathbf{r} = x(u,v,w)\hat{\mathbf{i}} + y(u,v,w)\hat{\mathbf{j}} + z(u,v,w)\hat{\mathbf{k}}$$

$$d\mathbf{r} = h_u du\hat{\mathbf{u}} \; + \; h_v dv\hat{\mathbf{v}} \; + \; h_w dw\hat{\mathbf{w}}$$

$$h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right| \qquad h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right| \qquad h_w = \left|\frac{\partial \mathbf{r}}{\partial w}\right|$$

$$\left|\frac{\partial \mathbf{r}}{\partial u}\right| = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right]^{1/2}$$

and similarly for others.

Surface integrals and curvilinear coordinates

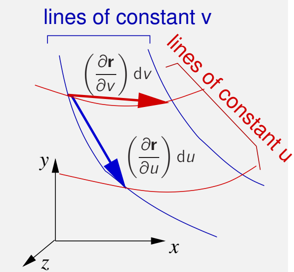
The surface element is a **vector product**

$$d\mathbf{S}_i = (dy\hat{\mathbf{j}}) \times (dz\hat{\mathbf{k}})$$

In curvi coords

$$d\mathbf{S}_w \neq (du\hat{\mathbf{u}}) \times (dv\hat{\mathbf{v}})$$

Locally the surface element is planar, so



$$\begin{aligned} d\mathbf{S}_w &= \frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv \\ &= h_u du\hat{\mathbf{u}} \times h_v dv\hat{\mathbf{v}} \end{aligned}$$

The general 3D result for $d\mathbf{S}_w$ in (u, v, w) coords is

$$d\mathbf{S}_w = h_u h_v du dv(\hat{\mathbf{u}} \times \hat{\mathbf{v}})$$

For an orthogonal curvilinear coord system, $\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{w}}$ and

$$d\mathbf{S}_w = h_u h_v du dv\hat{\mathbf{w}}$$

Surface integrals and curvilinear coordinates /ctd

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$$d\mathbf{S}_w = h_u h_v du dv \hat{\mathbf{w}}$$

In the $(x, y) \rightarrow (u, v)$ plane we arrive at a familiar result:

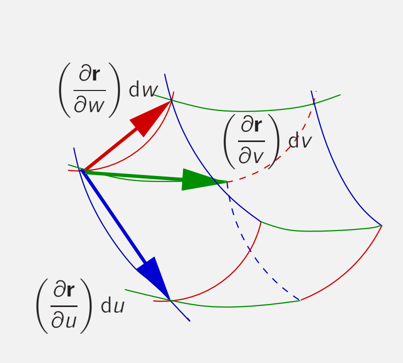
$$\begin{aligned} d\mathbf{S} &= \left(\frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}}\right) \times \left(\frac{\partial x}{\partial v}\hat{\mathbf{i}} + \frac{\partial y}{\partial v}\hat{\mathbf{j}}\right) du dv \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} du dv \\ &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du dv \ \hat{\mathbf{k}} \end{aligned}$$

Out pops the Jacobian!

Volume integrals and Curvilinear Coordinates

What is the size of the volume element in curvilinear coordinates?

It is the volume of a parallelopiped, which in an earlier lecture we saw was given by the **scalar triple product**.



Hence

$$dV = \left(\frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv\right) \cdot \frac{\partial \mathbf{r}}{\partial w} dw = h_u h_v h_w du dv dw (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}}$$

You can show that this is also the Jacobian: $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Volume integrals and Curvilinear Coordinates

To repeat:

$$dV = \left(\frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv\right) \cdot \frac{\partial \mathbf{r}}{\partial w} dw = h_u h_v h_w du dv dw (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}}$$

General 3D result

$$dV = h_u h_v h_w \; du \; dv \; dw \; (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}} = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \; du \; dv \; dw$$

Short cut if the system is orthogonal

$$dV = h_u h_v h_w \;\; du \; dv \; dw$$

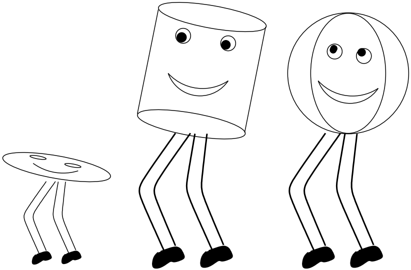
The Polars

Some curvilinear coordinate systems are orthogonal, meaning that $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are mutually perpendicular, so that

$$\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{w}} \qquad \text{and} \qquad (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}} = 1$$

We look at

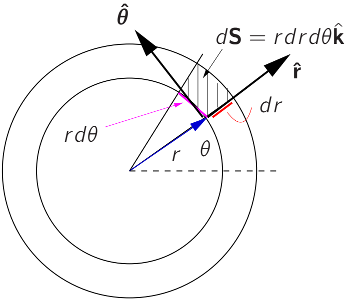
- plane polars
- cylindrical polars
- spherical polars



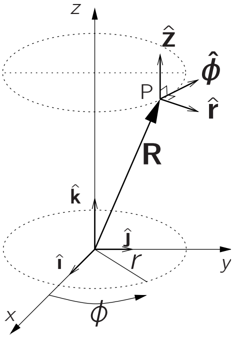
Plane polar co-ordinates

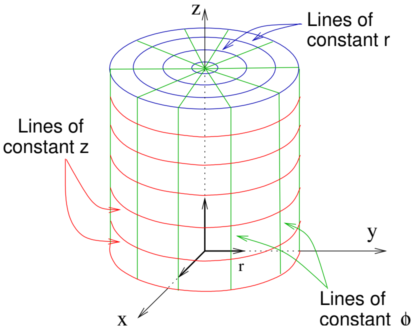
Start from the position vector:

$$
\begin{aligned}
\mathbf{r} &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}} \\
h_r\hat{\mathbf{r}} &= \frac{\partial \mathbf{r}}{\partial r} = (\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}) \\
h_\theta\hat{\theta} &= \frac{\partial \mathbf{r}}{\partial \theta} = (-r\sin\theta\hat{\mathbf{i}} + r\cos\theta\hat{\mathbf{j}}) \\
\Rightarrow h_r &= \left|\frac{\partial \mathbf{r}}{\partial r}\right| = |\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}| = 1 \\
h_\theta &= \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = |-r\sin\theta\hat{\mathbf{i}} + r\cos\theta\hat{\mathbf{j}}| = r \\
\hat{\mathbf{r}} &= (\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}) \\
\hat{\theta} &= (-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}) \\
\Rightarrow d\mathbf{r} &= h_r\, dr\, \hat{\mathbf{r}} + h_\theta\, d\theta\, \hat{\theta} = dr\, \hat{\mathbf{r}} + r\, d\theta\, \hat{\theta}. \\
\text{and } d\mathbf{S} &= h_r h_\theta\, dr\, d\theta\, (\hat{\mathbf{r}} \times \hat{\theta}) = r\, dr\, d\theta\, \hat{\mathbf{k}}\ .
\end{aligned}
$$



Cylindrical polar coordinates





$$x = r\cos\phi \ , \quad y = r\sin\phi \ , \quad z = z$$

Position vector $\mathbf{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = r\cos\phi\hat{\mathbf{i}} + r\sin\phi\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Why change the notation of position vector from $\mathbf{r}$ to $\mathbf{R}$?
If we did not, r would not equal $|\mathbf{r}|$, and $\hat{\mathbf{r}}$ would not be in same dirn as $\mathbf{r}$.
This could be confusing.

Cylindrical polars /ctd

$$
\begin{aligned}
\mathbf{R} &= r\cos\phi\hat{\mathbf{\imath}} + r\sin\phi\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}} \\
h_r\hat{\mathbf{r}} &= \partial\mathbf{R}/\partial r = (\cos\phi\hat{\mathbf{\imath}} + \sin\phi\hat{\mathbf{\jmath}}) \\
h_\phi\hat{\boldsymbol{\phi}} &= \partial\mathbf{R}/\partial\phi = (-r\sin\phi\hat{\mathbf{\imath}} + r\cos\phi\hat{\mathbf{\jmath}}) \\
h_z\hat{\mathbf{z}} &= \partial\mathbf{R}/\partial z = \hat{\mathbf{k}} \\
\Rightarrow h_r &= 1 \quad \text{and} \quad \hat{\mathbf{r}} = \cos\phi\hat{\mathbf{\imath}} + \sin\phi\hat{\mathbf{\jmath}} \\
h_\phi &= r \quad \text{and} \quad \hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{\imath}} + \cos\phi\hat{\mathbf{\jmath}} \\
h_z &= 1 \quad \text{and} \quad \hat{\mathbf{z}} = \hat{\mathbf{k}} \\
\Rightarrow d\mathbf{R} &= dr\ \hat{\mathbf{r}} + r d\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\mathbf{z}} \\
\text{and } d\mathbf{S}_r &= h_\phi h_z d\phi dz(\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}}) = r\ d\phi\ dz\ \hat{\mathbf{r}} \\
d\mathbf{S}_\phi &= h_z h_r dz dr(\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = dz\ dr\ \hat{\boldsymbol{\phi}} \\
d\mathbf{S}_z &= h_r h_\phi dr d\phi(\hat{\mathbf{r}} \times \hat{\mathbf{z}}) = r dr d\phi\hat{\mathbf{z}} \\
dV &= r\ dr\ d\phi\ dz
\end{aligned}
$$

♣ Example: Line integral in cylindrical polars

From the list: change in position vector is $d\mathbf{R} = dr\ \hat{\mathbf{r}} + r d\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\mathbf{z}}$

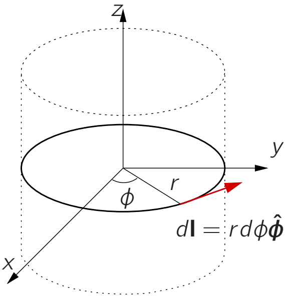
Q: Evaluate $\oint_C \mathbf{a} \cdot d\mathbf{R}$, where $\mathbf{a} = x^3\hat{\mathbf{j}} - y^3\hat{\mathbf{i}} + x^2 y\hat{\mathbf{k}}$ and C is the circle of radius ρ in the $z = 0$ plane, centred on the origin.

A: Turn **a** into cp's
$\mathbf{a} = \rho^3(-\sin^3\phi\hat{\mathbf{i}} + \cos^3\phi\hat{\mathbf{j}} + \cos^2\phi\sin\phi\hat{\mathbf{k}})$
Since $dz = dr = 0$ on our particular path,
and the constant $r = \rho$,

$$d\mathbf{R} = \rho\ d\phi\ \hat{\boldsymbol{\phi}} = \rho d\phi(-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}})$$

so that

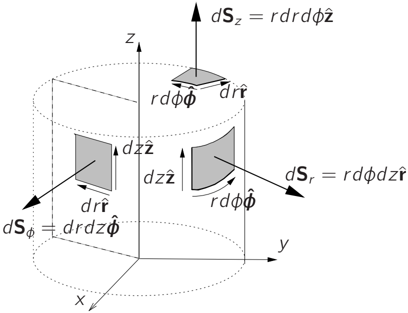
$$\oint_C \mathbf{a}\cdot d\mathbf{R} = \int_0^{2\pi} \rho^4(\sin^4\phi + \cos^4\phi)d\phi = \frac{3\pi}{2}\rho^4$$



NB! For line integrals you will often see the element along the path written as $d\ell$ (or $d\mathbf{r}$). Just roll with it ...

Surface integrals in cylindrical polars

Reminder ...



Cylindrical polars:
Often-used surface area elements are:

$$
\begin{aligned}
d\mathbf{S}_z &= dr\hat{\mathbf{r}} \times rd\phi\hat{\boldsymbol{\phi}} = r\,dr\,d\phi\,\hat{\mathbf{z}} \\
d\mathbf{S}_r &= rd\phi\hat{\boldsymbol{\phi}} \times dz\hat{\mathbf{z}} = r\,d\phi\,dz\,\hat{\mathbf{r}}
\end{aligned}
$$

Less often needed is

$$d\mathbf{S}_\phi = h_z h_r dzdr(\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = dz\ dr\ \hat{\boldsymbol{\phi}}$$

♣ Example: Surface integral in cyl polars

Q: Find $\int_S \mathbf{v} \cdot d\mathbf{S}$ when $\mathbf{v} = y^2\hat{\mathbf{\imath}} + x^2\hat{\mathbf{\jmath}}$ and the surface S is a cylinder of radius a and height h whose base sits on the x, y plane and whose axis coincides with $\hat{\mathbf{k}}$.

A: $\mathbf{v}$ has zero $\hat{\mathbf{k}}$ component, so there is no contribution from the top (where $d\mathbf{S} = +r\,dr\,d\phi\hat{\mathbf{k}}$) or bottom ($d\mathbf{S} = -r\,dr\,d\phi\hat{\mathbf{k}}$).
From the wall of the cylinder

$$\int \mathbf{v} \cdot d\mathbf{S} = \int_{z=0}^{h} \int_{\phi=0}^{2\pi} (a^2 \sin^2\phi\hat{\mathbf{\imath}} + a^2\cos^2\phi\hat{\mathbf{\jmath}}) \cdot (a\,d\phi\,dz\,\hat{\mathbf{r}})$$

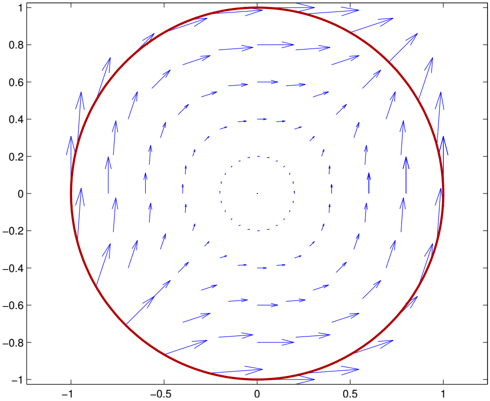
But $\hat{\mathbf{r}} = \cos\phi\hat{\mathbf{\imath}} + \sin\phi\hat{\mathbf{\jmath}}$, so

$$\begin{aligned}\int \mathbf{v} \cdot d\mathbf{S} &= a^3 \int_{z=0}^{h} \int_{\phi=0}^{2\pi} (\sin^2\phi\cos\phi + \cos^2\phi\sin\phi)\,d\phi\,dz \\ &= \frac{a^3 h}{3}\left[\sin^3\phi - \cos^3\phi\right]_0^{2\pi} = \boxed{0}\end{aligned}$$

Can we see why zero? ....

Surface integrals in cylindrical polars

... plot the vector field $\mathbf{v} = y^2\hat{\mathbf{i}} + x^2\hat{\mathbf{j}}$ from above. The red ring is the cylinder.



As much $\mathbf{v}$ flows in as flows out, and $\int \mathbf{v} \cdot d\mathbf{S}$ is the net outflow or efflux.

Surface integrals in cylindrical polars

Q: Find $\int_S \mathbf{v} \cdot d\mathbf{S}$ when $\mathbf{v} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}}$ and the surface S is a cylinder of radius a and height h whose base sits on the x, y plane and whose axis coincides with $\hat{\mathbf{k}}$.

A: $\mathbf{v}$ has zero $\hat{\mathbf{k}}$ component, hence there is no contribution from the top and bottom.
From the wall of the cylinder

$$\int \mathbf{v} \cdot d\mathbf{S} = \int_{z=0}^{h} \int_{\phi=0}^{2\pi} (a \cos\phi \hat{\mathbf{\imath}} + a \sin\phi \hat{\mathbf{\jmath}}) \cdot a \, d\phi \, dz \, \hat{\mathbf{r}}$$

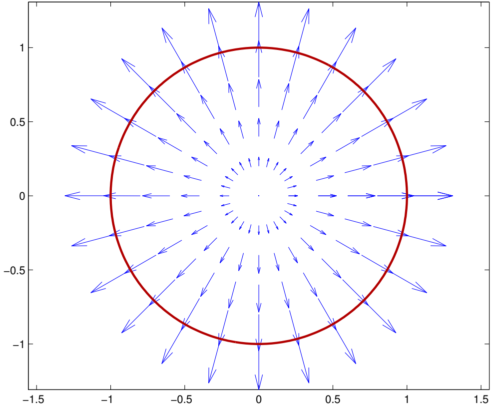
But $\hat{\mathbf{r}} = \cos\phi \hat{\mathbf{\imath}} + \sin\phi \hat{\mathbf{\jmath}}$, so

$$\begin{aligned} \int \mathbf{v} \cdot d\mathbf{S} &= a^2 \int_{z=0}^{h} \int_{\phi=0}^{2\pi} \ldots \\ &= \ldots. \\ &= 2\pi \, h \, a^2 \end{aligned}$$

Can we see why finite and positive? ...

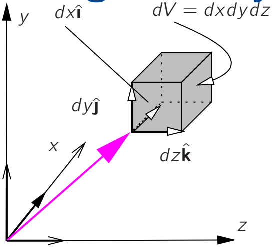
Surface integrals in cylindrical polars

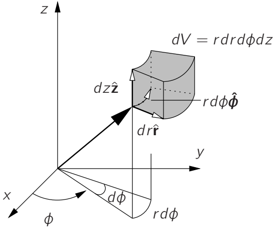
... plot the vector field $\mathbf{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$. The red ring is the cylinder ...



$\int \mathbf{v} \cdot d\mathbf{S}$ is the net efflux — clearly positive

Volume integrals in cylindrical polars





In Cartesians, volume element given by

$$dV = dx\hat{\mathbf{i}} \cdot (dy\hat{\mathbf{j}} \times dz\hat{\mathbf{k}}) = dx\,dy\,dz$$

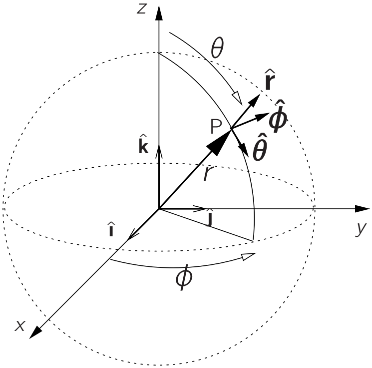
In cylindrical polars, volume element given by

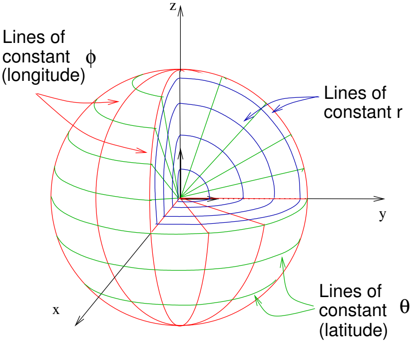
$$dV = dr\,\hat{\mathbf{r}} \cdot (rd\phi\hat{\boldsymbol{\phi}} \times dz\hat{\mathbf{z}}) = r\,d\phi\,dr\,dz$$

NB: Volume is scalar triple product, hence:

$$dV = \begin{vmatrix} \hat{\mathbf{r}}dr \\ \hat{\boldsymbol{\phi}}rd\phi \\ \hat{\mathbf{z}}dz \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} dr\,d\phi\,dz$$

Spherical polars





Can use $\mathbf{r}$ again ...

$$
\begin{aligned}
x &= r\sin\theta\cos\phi\ , \quad y = r\sin\theta\sin\phi\ , \quad z = r\cos\theta \\
\mathbf{r} &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \\
&= r\sin\theta\cos\phi\hat{\mathbf{i}} + r\sin\theta\sin\phi\hat{\mathbf{j}} + r\cos\theta\hat{\mathbf{k}}
\end{aligned}
$$

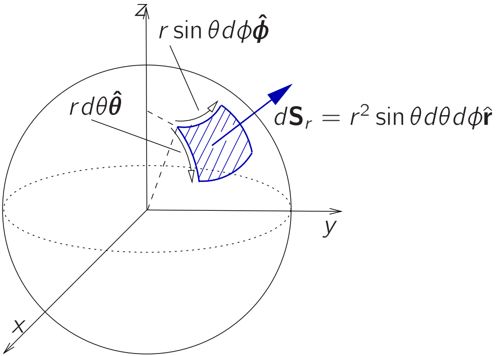
Spherical polars /ctd

$$
\begin{aligned}
\mathbf{r} &= r\sin\theta\cos\phi\hat{\mathbf{i}} + r\sin\theta\sin\phi\hat{\mathbf{j}} + r\cos\theta\hat{\mathbf{k}} \\
\Rightarrow h_r\hat{\mathbf{r}} &= \partial\mathbf{r}/\partial r = \\
h_\theta\hat{\boldsymbol{\theta}} &= \partial\mathbf{r}/\partial\theta = \\
h_\phi\hat{\boldsymbol{\phi}} &= \partial\mathbf{r}/\partial\phi = \\
\Rightarrow h_r &= 1\ ,\quad h_\theta = r\sin\theta,\quad h_\phi = r \\
\Rightarrow \hat{\mathbf{r}} &= \sin\theta\ \cos\phi\hat{\mathbf{i}} + \sin\theta\ \sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\ \cos\phi\hat{\mathbf{i}} + \cos\theta\ \sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}} \\
\hat{\boldsymbol{\phi}} &= -\sin\theta\hat{\mathbf{i}} + \cos\phi\ \hat{\mathbf{j}} \\
\Rightarrow d\mathbf{r} &= dr\ \hat{\mathbf{r}} + r d\theta\ \hat{\boldsymbol{\theta}} + r\sin\theta\ d\phi\ \hat{\boldsymbol{\phi}} \\
d\mathbf{S}_r &= r^2\ \sin\theta\ d\theta\ d\phi\ \hat{\mathbf{r}} \qquad \text{on spherical surface} \\
d\mathbf{S}_\theta &= \boxed{?}\ dr\ d\theta\ \hat{\boldsymbol{\theta}} \qquad \text{on conical surface: DIY} \\
d\mathbf{S}_\phi &= r\ dr\ d\phi\ \hat{\boldsymbol{\phi}} \qquad \text{on planar hemisphere surface} \\
dV &= r^2\ \sin\theta\ dr\ d\theta\ d\phi
\end{aligned}
$$

Surface integrals in spherical polars

Three possibilities, but most useful are surfaces of constant r
The surface element $d\mathbf{S}_r$ is given by

$$\begin{aligned} d\mathbf{S}_r &= h_\theta d\theta\hat{\boldsymbol{\theta}} \times h_\phi d\phi\hat{\boldsymbol{\phi}} \\ &= r^2 \sin\theta d\theta d\phi\hat{\mathbf{r}} \end{aligned}$$



♣ Example: Surface integral in spherical polars

Q: Evaluate $\int_S \mathbf{a} \cdot d\mathbf{S}$, where $\mathbf{a} = z^3 \hat{\mathbf{k}}$ and S is the sphere of radius A centred on the origin.

A: In general:

$$z = r\cos\theta \qquad d\mathbf{S} = r^2 \sin\theta d\theta d\phi \hat{\mathbf{r}}$$

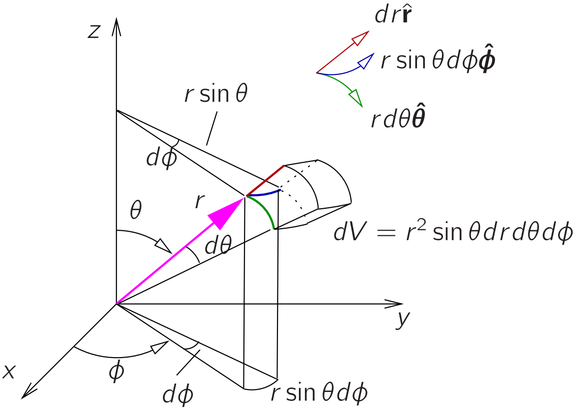
On surface of the sphere, $r = A$, so that

$$\mathbf{a} = z^3 \hat{\mathbf{k}} = A^3 cos^3\theta \hat{\mathbf{k}} \qquad d\mathbf{S} = A^2 \sin\theta \; d\theta \; d\phi \hat{\mathbf{r}}$$

Hence

$$\begin{aligned} \int_S \mathbf{a} \cdot d\mathbf{S} &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} A^3 cos^3\theta \; A^2 \sin\theta \; [\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}] \; d\theta d\phi \\ &= A^5 \int_0^{2\pi} d\phi \; \int_0^{\pi} cos^3\theta \sin\theta [\cos\theta] \; d\theta \\ &= 2\pi A^5 \frac{1}{5} \left[-\cos^5\theta\right]_0^{\pi} = \frac{4\pi A^5}{5} \end{aligned}$$

Volume integrals in spherical polars



- Volume element given by

$$dV = dr\hat{\mathbf{r}}.(rd\theta\hat{\boldsymbol{\theta}} \times r\sin\theta d\phi\hat{\boldsymbol{\phi}}) = r^2\sin\theta dr d\theta d\phi$$

- Note again that this volume could be written as a determinant

Summary

- We introduced line, surface and volume integrals involving vector fields.
- We defined *curvilinear coordinates*, and realized that metric coefficient were necessary to relate change in an arbitrary coordinate to a length scale.
- We showed in detail how line, surface and volume elements are derived, and how the results specialized for orthogonal curvilinear system, in particular plane, cylindrical and spherical polar coordinates.
- Working stuff out from first principles has been hard going: as the examples showed, application is much easier!