A1 Vector Algebra and Calculus

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Vector Algebra and Calculus

1. Revision of vector algebra, scalar product, vector product
2. **Triple products, multiple products, applications to geometry**
3. Differentiation of vector functions, applications to mechanics
4. Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates
5. Vector operators — grad, div and curl
6. Vector Identities, curvilinear co-ordinate systems
7. Gauss' and Stokes' Theorems and extensions
8. Engineering Applications

More Algebra & Geometry using Vectors

In which we discuss ...

- Vector products:
  Scalar Triple Product, Vector Triple Product, Vector Quadruple Product
- Geometry of Lines and Planes
- Solving vector equations
- Angular velocity and moments

Triple and multiple products

Using mixtures of scalar products and vector products, it is possible to derive

- "triple products" between three vectors
- n-products between n vectors.

Nothing new about these
— but some have nice geometric interpretations ...

We will look at the

- Scalar triple product
- Vector triple product
- Vector quadruple product

Scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

Scalar triple product given by the true determinant

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Your knowledge of determinants tells you that if you

- swap one pair of rows of a determinant, sign changes;
- swap two pairs of rows, its sign stays the same.

Hence

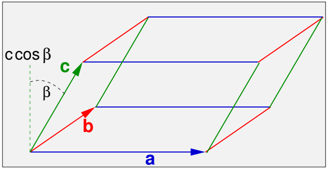
(i) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ (Cyclic permutation.)

(ii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$ and so on. (Anti-cyclic permutation)

(iii) The fact that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ allows the scalar triple product to be written as $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.
This notation is not very helpful, and we will try to avoid it below.

Geometrical interpretation of scalar triple product

The scalar triple product gives the volume of the parallelopiped whose sides are represented by the vectors **a**, **b**, and **c**.



Vector product $(\mathbf{a} \times \mathbf{b})$ has magnitude equal to the area of the base $\times$ height in direction perpendicular to the base.

The *component* of **c** in this direction is equal to the height of the parallelopiped, hence

$$\text{volume of parallelopiped} = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$$

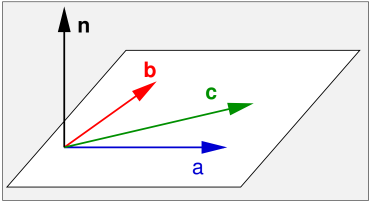
Linearly dependent vectors

If the scalar triple product of three vectors

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

then the vectors are **linearly dependent**.

$$\mathbf{a} = \lambda \mathbf{b} + \mu \mathbf{c}$$



You can see this immediately either using the determinant

— The determinant would have one row that was a linear combination of the others

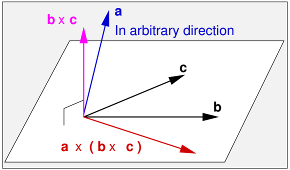
or geometrically for a 3-dimensional vector.

— the parallelopiped would have zero volume if squashed flat.

Vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to $(\mathbf{b} \times \mathbf{c})$
but $(\mathbf{b} \times \mathbf{c})$ is perpendicular to $\mathbf{b}$ *and* $\mathbf{c}$.
So $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ must be *coplanar* with $\mathbf{b}$ and $\mathbf{c}$.

$$\Rightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \lambda \mathbf{b} + \mu \mathbf{c}$$



$$\begin{aligned}
(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_1 &= a_2(\mathbf{b} \times \mathbf{c})_3 - a_3(\mathbf{b} \times \mathbf{c})_2 \\
&= a_2(b_1 c_2 - b_2 c_1) + a_3(b_1 c_3 - b_3 c_1) \\
&= (a_2 c_2 + a_3 c_3) b_1 - (a_2 b_2 + a_3 b_3) c_1 \\
&= (a_1 c_1 + a_2 c_2 + a_3 c_3) b_1 - (a_1 b_1 + a_2 b_2 + a_3 b_3) c_1 \\
&= (\mathbf{a} \cdot \mathbf{c}) b_1 - (\mathbf{a} \cdot \mathbf{b}) c_1
\end{aligned}$$

Similarly for components 2 and 3, so

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

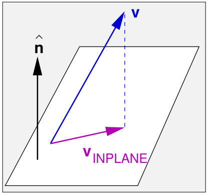
Projection using vector triple product

Books say that the vector projection of any vector $\mathbf{v}$ into a plane with normal $\hat{\mathbf{n}}$ is

$$\mathbf{v}_{\text{IN PLANE}} = \hat{\mathbf{n}} \times (\mathbf{v} \times \hat{\mathbf{n}})$$

We would say that the component of $\mathbf{v}$ in the $\hat{\mathbf{n}}$ direction is $\mathbf{v} \cdot \hat{\mathbf{n}}$, so the vector projection is

$$\mathbf{v}_{\text{IN PLANE}} = \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$$



Can we reconcile the two expressions? (Yes we can.)
Subst. $\hat{\mathbf{n}} \leftarrow \mathbf{a}$, $\mathbf{v} \leftarrow \mathbf{b}$, $\hat{\mathbf{n}} \leftarrow \mathbf{c}$, into our earlier formula

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\ \hat{\mathbf{n}} \times (\mathbf{v} \times \hat{\mathbf{n}}) &= (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\mathbf{v} - (\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}} \\ &= \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \end{aligned}$$

Fantastico! But $\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$ is much easier to understand ...

... and cheaper to compute!

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Vector Quadruple Product $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$

We have just learned that

$$\begin{aligned} \mathbf{p}\times(\mathbf{q}\times\mathbf{r}) &= (\mathbf{p}\cdot\mathbf{r})\mathbf{q}-(\mathbf{p}\cdot\mathbf{q})\mathbf{r} \\ \Rightarrow\quad (\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= ?? \end{aligned}$$

Regarding $\mathbf{a}\times\mathbf{b}$ as a single vector
$\Rightarrow$ vqp must be a linear combination of $\mathbf{c}$ and $\mathbf{d}$

Regarding $\mathbf{c}\times\mathbf{d}$ as a single vector
$\Rightarrow$ vqp must be a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Substituting in carefully (you check ...)

$$\begin{aligned} (\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}]\mathbf{c}-[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]\mathbf{d} \\ \text{and also} &= [(\mathbf{c}\times\mathbf{d})\cdot\mathbf{a}]\mathbf{b}-[(\mathbf{c}\times\mathbf{d})\cdot\mathbf{b}]\mathbf{a} \end{aligned}$$

Vector Quadruple Product /ctd

Using just the R-H sides of what we just wrote ...

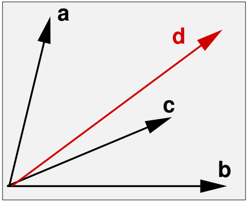
$$[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]\,\mathbf{d} = [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{d}]\,\mathbf{a} + [(\mathbf{c}\times\mathbf{a})\cdot\mathbf{d}]\,\mathbf{b} + [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}]\,\mathbf{c}$$

So

$$\begin{aligned}\mathbf{d} &= \frac{[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{d}]\,\mathbf{a} + [(\mathbf{c}\times\mathbf{a})\cdot\mathbf{d}]\,\mathbf{b} + [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}]\,\mathbf{c}}{[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]} \\ &= \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \quad .\end{aligned}$$

Oh, we saw this yesterday ... ... the

projection of a 3D vector **d** onto a basis set of 3 non-coplanar vectors is UNIQUE.



♣ Example

Q:
Use the quadruple vector product to express the vector $\mathbf{d} = [3, 2, 1]$ in terms of the vectors $\mathbf{a} = [1, 2, 3]$, $\mathbf{b} = [2, 3, 1]$ and $\mathbf{c} = [3, 1, 2]$.
A:

$$\mathbf{d} = \frac{[(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\,\mathbf{a} + [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}]\,\mathbf{b} + [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\,\mathbf{c}}{[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]}$$

So, grinding away at the determinants, we find

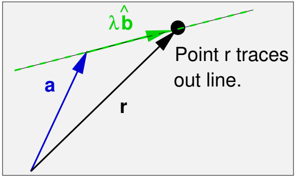
- $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -18$ and $(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d} = 6$
- $(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d} = -12$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} = -12$.

So

$$\begin{aligned} \mathbf{d} &= \frac{1}{-18}(6\mathbf{a} - 12\mathbf{b} - 12\mathbf{c}) \\ &= \frac{1}{3}(-\mathbf{a} + 2\mathbf{b} + 2\mathbf{c}) \end{aligned}$$

Geometry using vectors: Lines

Equation of line passing through point $\mathbf{a}_1$ and lying in the direction of vector $\mathbf{b}$ is



$$\mathbf{r} = \mathbf{a} + \beta \mathbf{b}$$

NB! Only when you make a unit vector in the dirn of $\mathbf{b}$ does the parameter take on the length units defined by $\mathbf{a}$:

$$\mathbf{r} = \mathbf{a} + \lambda \hat{\mathbf{b}}$$

For a line defined by two points $\mathbf{a}_1$ and $\mathbf{a}_2$

$$\mathbf{r} = \mathbf{a}_1 + \beta(\mathbf{a}_2 - \mathbf{a}_1)$$

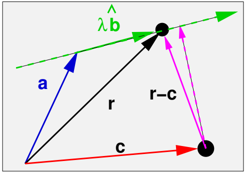
or the unit version ...

$$\mathbf{r} = \mathbf{a}_1 + \lambda(\mathbf{a}_2 - \mathbf{a}_1)/|\mathbf{a}_2 - \mathbf{a}_1|$$

The shortest distance from a point to a line

Vector **p** from **c** to ANY line point **r** is

$$\mathbf{p} = (\mathbf{r} - \mathbf{c}) = \mathbf{a} + \lambda\hat{\mathbf{b}} - \mathbf{c} = (\mathbf{a} - \mathbf{c}) + \lambda\hat{\mathbf{b}}$$



which has length squared

$$p^2 = (\mathbf{a} - \mathbf{c})^2 + \lambda^2 + 2\lambda(\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}} \ .$$

Easier to minimize p^2 rather than p itself.

$$\frac{d}{d\lambda}p^2 = 0 \quad \text{when} \quad \lambda = -(\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}} \ .$$

So the minimum length vector is $\mathbf{p} = (\mathbf{a} - \mathbf{c}) - [(\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}}]\hat{\mathbf{b}}$.

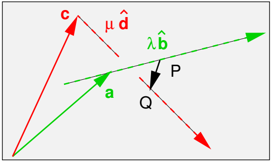
No surprise! It's the component of $(\mathbf{a} - \mathbf{c})$ **perpendicular** to $\hat{\mathbf{b}}$.

Shortest distance between two straight lines

Shortest distance from point to line is along the perp line

$\Rightarrow$ shortest distance between 2 lines is along mutual perpendicular.

The lines are:
$\mathbf{r} = \mathbf{a} + \lambda\hat{\mathbf{b}}$ $\quad$ $\mathbf{r} = \mathbf{c} + \mu\hat{\mathbf{d}}$



The unit vector along the mutual perp is

$$\hat{\mathbf{p}} = \frac{\hat{\mathbf{b}} \times \hat{\mathbf{d}}}{|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|} \ .$$

(Yes! Don't forget that $\hat{\mathbf{b}} \times \hat{\mathbf{d}}$ is NOT a unit vector.)

The minimum length is therefore the component of $(\mathbf{a} - \mathbf{c})$ in this direction

$$p_{\text{min}} = \left| (\mathbf{a} - \mathbf{c}) \cdot \left(\frac{\hat{\mathbf{b}} \times \hat{\mathbf{d}}}{|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|} \right) \right| \ .$$

♣ Example

Q: for civil engineers who like pipes
Two long straight pipes are specified using Cartesian co-ordinates as follows:

Pipe A: diameter 0.8; axis through points $(2, 5, 3)$ and $(7, 10, 8)$.

Pipe B: diameter 1.0; axis through points $(0, 6, 3)$ and $(-12, 0, 9)$.

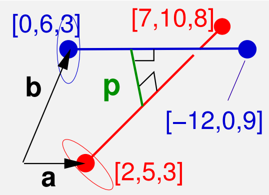
Do the pipes need re-aligning to avoid intersection?

♣ /ctd

A: Pipes A and B have axes:

$$\mathbf{r}_A = [2,5,3] + \lambda'[5,5,5] = [2,5,3] + \lambda[1,1,1]/\sqrt{3}$$
$$\mathbf{r}_B = [0,6,3] + \mu'[-12,-6,6] = [0,6,3] + \mu[-2,-1,1]/\sqrt{6}$$

Non-unit perpendicular to both axes is



$$\mathbf{p} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = [2,-3,1]$$

The length of the mutual perpendicular is

$$\left|(\mathbf{a}-\mathbf{b})\cdot\frac{[2,-3,1]}{\sqrt{14}}\right| = \frac{[2,-1,0]\cdot[2,-3,1]}{\sqrt{14}} = 1.87\ .$$

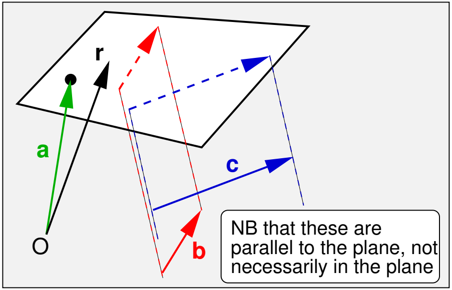
Sum of the radii of the pipes is $0.4 + 0.5 = 0.9$. $\Rightarrow$**no collision**

Three ways of describing a plane. Number 1

1. Point + 2 non-parallel vectors
If **b** and **c** non-parallel, and **a** is a point on the plane, then

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

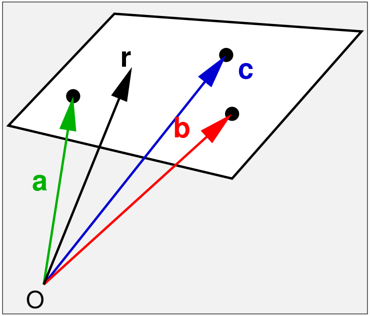
where λ, μ are scalar parameters.



Three ways of describing a plane. Number 2

2. Three points
Points **a**, **b** and **c** in the plane.

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

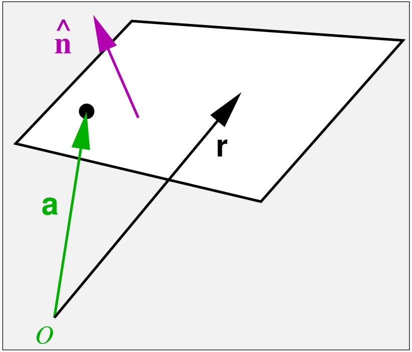


Vectors $(\mathbf{b} - \mathbf{a})$ and $(\mathbf{c} - \mathbf{a})$ are said to **span the plane**.

Three ways of describing a plane. Number 3

3. Unit normal Unit normal to the plane is $\hat{\mathbf{n}}$, and a point in the plane is $\mathbf{a}$

$$\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}} = D$$



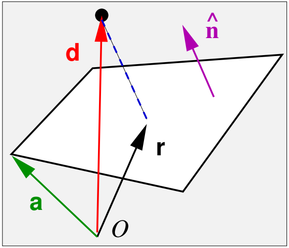
Notice that $|D|$ is the perpendicular distance to the plane from the origin.

The shortest distance from a point to a plane

The plane is $\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}} = D$

Now, the shortest distance from point $\mathbf{d}$ to the plane ... ?

1. Must be along the perpendicular
2. $\mathbf{d} + \lambda \hat{\mathbf{n}}$ must be a point on plane

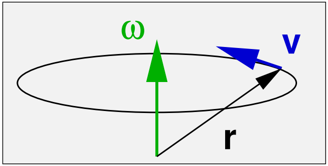


$$\Rightarrow \quad (\mathbf{d} + \lambda \hat{\mathbf{n}}) \cdot \hat{\mathbf{n}} = D$$
$$\Rightarrow \quad \lambda = D - \mathbf{d} \cdot \hat{\mathbf{n}}$$
$$\Rightarrow \quad d_{min} = |\lambda| = |D - \mathbf{d} \cdot \hat{\mathbf{n}}|$$

Rotation, angular velocity and acceleration

A rotation can represented by a vector whose

- direction is along the axis of rotation in the sense of a right-handed screw,
- magnitude is proportional to the size of the rotation.



The same idea can be extended to the derivatives

- angular velocity $\boldsymbol{\omega}$
- angular acceleration $\dot{\boldsymbol{\omega}}$.

The instantaneous velocity $\mathbf{v}(\mathbf{r})$ of any point P at $\mathbf{r}$ on a rigid body undergoing pure rotation can be defined by a vector product

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

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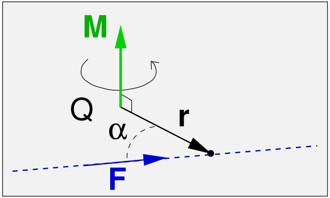
Vector Moments

Angular accelerations arise because of moments.

The vector equation for the moment **M** of a force **F** about a point Q is

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

where **r** is a vector from Q to *any* point on the line of action L of force **F**.



The resulting angular acceleration $\dot{\boldsymbol{\omega}}$ is in the same direction as the moment vector **M**. (How are they related?)

Solution of vector equations

Find the most general vector $\mathbf{x}$ satisfying a given vector relationship. Eg

$$\mathbf{x} = \mathbf{x} \times \mathbf{a} + \mathbf{b}$$

General Method (assuming 3 dimensions)

1. Set up a system of **three basis vectors** using **two** non-parallel vectors appearing in the original vector relationship. For example
$$\mathbf{a},\ \mathbf{b},\ (\mathbf{a} \times \mathbf{b})$$
2. Write
$$\mathbf{x} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{a} \times \mathbf{b}$$
where λ, μ, ν are scalars to be found.
3. Substitute expression for $\mathbf{x}$ into the vector relationship to determine the set of constraints on λ, μ, and ν.

♣ Example 1: Solve $\mathbf{x} = (\mathbf{x} \times \mathbf{a}) + \mathbf{b}$.

Step 1: Basis vectors **a**, **b** and vector product $\mathbf{a} \times \mathbf{b}$.
Step 2: $\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{a} \times \mathbf{b}$.
Step 3: Stick **x** back into the equation ...

$$\begin{aligned} \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{a} \times \mathbf{b} &= \big(\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{a} \times \mathbf{b}\big) \times \mathbf{a} + \mathbf{b} \\ &= \mathbf{0} + \mu(\mathbf{b} \times \mathbf{a}) + \nu(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} \quad + \quad \mathbf{b} \end{aligned}$$

But $(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} = a^2\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a}$

$$\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{a} \times \mathbf{b} = -\nu(\mathbf{a} \cdot \mathbf{b})\mathbf{a} + (\nu a^2 + 1)\mathbf{b} - \mu(\mathbf{a} \times \mathbf{b})$$

Equating coefficients of **a**, **b** and $\mathbf{a} \times \mathbf{b}$ in the equation gives

$$\lambda = -\nu(\mathbf{a} \cdot \mathbf{b}) \qquad \mu = \nu a^2 + 1 \qquad \nu = -\mu$$

$$\Rightarrow \quad \mu = 1/(1 + a^2) \qquad \nu = -1/(1 + a^2) \qquad \lambda = (\mathbf{a} \cdot \mathbf{b})(1 + a^2)\ .$$

So finally the solution is the single point:

$$\mathbf{x} = \frac{1}{1 + a^2}[(\mathbf{a} \cdot \mathbf{b})\mathbf{a} + \mathbf{b} - (\mathbf{a} \times \mathbf{b})]$$

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♣ Example 2: Solve $\mathbf{x} \cdot \mathbf{a} = K$

This is in 2A1A, but we want to think around it ...
First note that there are not two fixed vectors in the expression ...
A:
Step 1 Use **a**, and introduce an arbitrary vector **b**, then find $\mathbf{a} \times \mathbf{b}$
Step 2: $\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{a} \times \mathbf{b}$.
Step 3: Bung **x** back into the equation!

... GRIND AWAY ...

and, recalling λ and ν are free parameters, we find

$$\mathbf{x} = \lambda\mathbf{a} + \left[\frac{K - \lambda a^2}{\mathbf{b} \cdot \mathbf{a}}\right]\mathbf{b} + \nu\mathbf{a} \times \mathbf{b}$$

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♣ Example #2: $\mathbf{x} \cdot \mathbf{a} = K$

$$\mathbf{x} = \lambda \mathbf{a} + \left[\frac{K - \lambda a^2}{\mathbf{b} \cdot \mathbf{a}} \right] \mathbf{b} + \nu \mathbf{a} \times \mathbf{b}$$

This is certainly correct ... but it looks very odd, given that the geometry is very obvious in this case!

$\mathbf{x}$ must lie on the plane $\mathbf{x} \cdot \hat{\mathbf{a}} = K/a$ ...
... a plane with unit normal $\hat{\mathbf{a}}$ and perpendicular distance $|K/a|$ from the origin.

So why does it look so complicated?

It is because $\mathbf{b}$ has been chosen arbitrarily *and* is one of the basis vectors.

♣ Example 2: $\mathbf{x} \cdot \mathbf{a} = K$

As we can see upfront that this must be a plane,
here is a cunning plan ...

Choose **b** arbitrarily, but don't use **b** as the second vector

Instead use it to find a second vector that is
perpendicular to BOTH **a** AND $(\mathbf{a} \times \mathbf{b})$.

We can write down **without further thought**

$$\mathbf{x} = \frac{K}{a^2}\mathbf{a} + \mu(\mathbf{a} \times (\mathbf{a} \times \mathbf{b})) + \nu(\mathbf{a} \times \mathbf{b}) \; . \qquad \mu, \nu \text{ are free}$$

Can you see why?

A comment about solving vector identities

Suppose you are faced with

$$\mu \mathbf{a} + \lambda \mathbf{b} = \mathbf{c}$$

and you want to find μ.

What is the fast way of getting rid of $\mathbf{b}$?

Use $(\mathbf{b} \times \mathbf{b}) = \mathbf{0}$ ...

$$\begin{aligned} \mu(\mathbf{a} \times \mathbf{b}) &= \mathbf{c} \times \mathbf{b} \\ \Rightarrow \mu(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) &= (\mathbf{c} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) \\ \Rightarrow \mu &= \frac{(\mathbf{c} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})}{(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})} \end{aligned}$$

A comment about solving vector identities

$\mu\mathbf{a} + \lambda\mathbf{b} = \mathbf{c}$

An alternative is to construct two simultaneous equations

$$\begin{aligned}\mu\mathbf{a}\cdot\mathbf{b} + \lambda b^2 &= \mathbf{c}\cdot\mathbf{b}\\ \mu a^2 + \lambda\mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot\mathbf{c}\end{aligned}$$

and eliminate λ

$$\mu = \frac{(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{c})b^2}{(\mathbf{a}\cdot\mathbf{b})^2 - a^2b^2}$$

Compare with previous

$$\mu = \frac{(\mathbf{c}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}$$

Summary

We've discussed ...

- Vector products
- Angular velocity/moments
- Line & Plane geometry
- Solving vector equations

Key point from Lectures 1 and 2:

☛ Use vectors and their algebra "constructively" to solve problems. (The elastic collision was a good example.)

☛ Don't be afraid to produce solutions that involve vector operations. Eg: $\mu = \mathbf{a} \cdot \mathbf{b} / |\mathbf{c} \times \mathbf{a}|$. Working out detail could be left to a computer program.

☛ Run with natural coordinate systems.

☛ If you are constantly breaking vectors into their components, you are (probably) not using their power.

☛ Apply checks that equations are vector or scalar on both sides. (Underline vectors.)