

# A1 Vector Algebra and Calculus

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8 lectures, MT 2015



# Vector Calculus & Scalar Fields

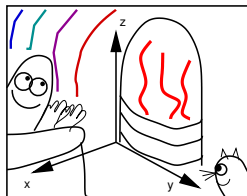
- Think about some **scalar quantities**
  - mass  $M$ , length  $L$ , time  $t$ , temperature  $T$ , etc
- If  $\mathbf{r} = [x, y, z]$  is a position in space,  $T(\mathbf{r})$  is a **scalar field**
- $T$  might be time-varying — the field is  $T(\mathbf{r}, t)$
- Keep  $y, z, t$  constant. What is  $\delta T$  when you move  $\delta x$ ?

$$\delta T = \left( \frac{\partial T}{\partial x} \right) \delta x.$$

- But suppose you moved off in a direction  $\mathbf{n}$ . Would you multiply

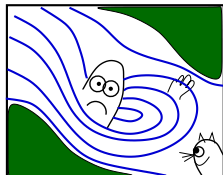
$$\delta T = \frac{\partial T}{\partial \mathbf{n}} \delta \mathbf{n} ?$$

- Does  $\partial T / \partial \mathbf{n}$  exist — is it a vector or a scalar?



# Vector Calculus & Vector Fields

- A vector quantity  $\mathbf{v}(\mathbf{r})$  that has a value at every  $\mathbf{r}$  in a region is a **vector field**.
- Examples are:
  - The electric field  $\mathbf{E}(\mathbf{r})$  around stationary charges
  - The unsteady fluid velocity field  $\mathbf{v}(\mathbf{r}, t)$  in a stream.
- Local stream velocity  $\mathbf{v}(\mathbf{r}, t)$  can be viewed using:
  - *laser Doppler anemometry*, or by dropping twigs in, or diving in ...
- You'll be interested in
  - weirs (acceleration), &
  - vortices (curls)



# Contents

- 1 Revision of vector algebra, scalar product, vector product.
- 2 Triple products, multiple products, applications to geometry.
- 3 Differentiation of vector functions, applications to mechanics.
- 4 Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates .
- 5 Vector operators — grad, div and curl.
- 6 Vector Identities, curvilinear co-ordinate systems.
- 7 Gauss' and Stokes' Theorems and extensions.
- 8 Engineering Applications.

# Learning Outcomes

- comfort with expressing systems using vector quantities
- manipulating vectors as “atomic” entities without recourse to underlying coordinates
- sound grasp of the concept of a vector field
- ability to link this idea to descriptions various physical phenomena
- intuition of the physical meaning of the various vector calculus operators (div, grad, curl)
- ability to interpret the formulae describing physical systems in terms of these operators

# Reading

- J Heading, "Mathematical Methods in Science and Engineering", 2nd ed., Ch.13, (Arnold).
- G Stephenson, "Mathematical Methods for Science Students", 2nd ed., Ch.19, (Longman).
- E Kreyszig, "Advanced Engineering Mathematics", 6th ed., Ch.6, (Wiley).
- K F Riley, M. P. Hobson and S. J. Bence, "Mathematical Methods for the Physics and Engineering" Chs. 6, 8 and 9 (CUP).
- A J M Spencer, et. al. "Engineering Mathematics", Vol.1, Ch.6, (Van Nostrand Reinhold).
- H M Schey, "Div, Grad, Curl and all that", Norton

# Course WWW Pages

- Pdf copies of
  - these oheads
  - lecture notes (also large print),
  - tutorial sheets (also large print)
  - FAQs etc

will be accessible from

**[www.robots.ox.ac.uk/~dwm/Courses/2VA](http://www.robots.ox.ac.uk/~dwm/Courses/2VA)**

- If something is *really* not clear, and you are *really* stuck,
  - email [david.murray@eng.ox.ac.uk](mailto:david.murray@eng.ox.ac.uk)

and the reply (if generally useful) will get stuck on the web FAQs.

# A1 Vector Algebra and Calculus

- 1 Revision of vector algebra, scalar product, vector product**
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# Vector Algebra

In which we explore ...

- Free, sliding and position vectors
- Coord frames and Vector components
- Equality, magnitude, Addition, Subtraction
- Scalar products, Vector Projection, Inner products
- Vector Products

## But first, some notation

We will use bold font for vectors  $\mathbf{a}, \mathbf{b} \dots \boldsymbol{\alpha}, \boldsymbol{\beta} \dots$   
and non-bold to for their magnitudes  $a, b \dots \alpha, \beta \dots$

In written work, underline your vectors.  
It is a way of avoiding chaos.

Woolly hats are worn by unit vectors  $\hat{\mathbf{a}}, \hat{\mathbf{b}} \dots \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}} \dots$

Here are some trios of unit vectors we will use

$$(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}) \quad (\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}}) \quad (\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}) \quad (\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$$

$\mathbf{a}^T$  denotes the transpose of a vector

mag and dirn are occasional shorthands for magnitude and direction

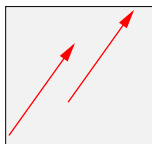
# Vectors

In linear algebra, vectors can be treated as abstract lists of numbers.

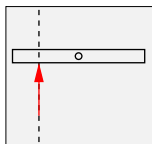
In this course we are concerned with vectors as representations of physical quantities that possess magnitude and direction.

There are three slightly different types of vectors:

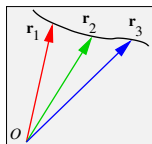
- **Free vectors:** *Only mag & dirn are important. Can translate at will.*
- **Sliding vectors:** Line of action is important (eg. forces for moments). Vectors can slide with 1 degree of freedom.
- **Bound or position vectors:** “Tails” all originate at origin  $O$ .



Free vectors



Sliding vectors

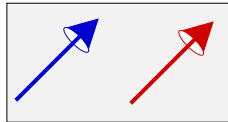


Position vectors

# Coordinate frames

An advantage of vector algebra:  
— frees analysis from arbitrarily imposed coordinate frames.

Eg, two free vectors are equal if mags and dirns are equal. Can be done with a drawing **independent of any coordinate system.**



Try to spot things in the notes that are independent of coordinate system.

However, coordinate systems are useful,  
so introduce the idea of **vector components.**

# Vector components in a coordinate frame

In a Cartesian coordinate frame

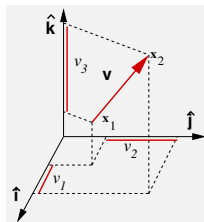
$$\mathbf{v} = [v_1, v_2, v_3] = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$$

Define  $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$  as unit vectors in the  $x, y, z$  dirns

$$\hat{\mathbf{i}} = [1, 0, 0] \quad \hat{\mathbf{j}} = [0, 1, 0] \quad \hat{\mathbf{k}} = [0, 0, 1]$$

then

$$\mathbf{v} = v_1 \hat{\mathbf{i}} + v_2 \hat{\mathbf{j}} + v_3 \hat{\mathbf{k}} .$$



# Vector equality

Two **free vectors** are said to be equal iff their lengths and directions are the same.

Using coordinates, two  $n$ -dimensional vectors are equal

$$\mathbf{a} = \mathbf{b} \quad \text{iff} \quad a_1 = b_1, \quad a_2 = b_2, \quad \dots \quad a_n = b_n$$

This applies to **position vectors** too.

But for **sliding vectors** we must add  
*the line of action must be the same.*

# Vector magnitude and unit vectors

Provided we use an orthogonal coordinate system, the magnitude of a 3-vector is

$$a = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

and of an  $n$ -vector

$$a = |\mathbf{a}| = \sqrt{\sum_i a_i^2}$$

To find the unit vector in the direction of  $\mathbf{a}$ , simply divide the vector by its magnitude

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} .$$

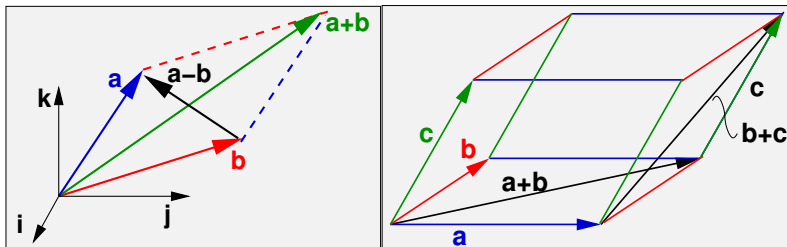
# Vector Addition and Subtraction

Vector addition follows the parallelogram construction (coord free), but if you like components ...

Vectors are added by adding corresponding components

$$\mathbf{a} + \mathbf{b} = [a_1 + b_1, \quad a_2 + b_2, \quad a_3 + b_3]$$

Subtraction is  $\mathbf{a} + (-\mathbf{b})$





# Properties of addition/subtraction

The following results follow immediately from the above definition of vector addition (incl. subtraction).

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$   
(it commutes)

- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = \mathbf{a} + \mathbf{b} + \mathbf{c}$   
(it associates)

- $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$   
where the zero vector is  $\mathbf{0} = [0, 0, 0]$ .

- $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

# Multiplication of a vector by a scalar

**NOT the scalar product!**

Just as for matrices, multiplication of a vector  $\mathbf{a}$  by a scalar  $c$  is defined as multiplication of each component by  $c$ , so that

$$c\mathbf{a} = [ca_1, ca_2, ca_3].$$

It follows that:

$$|c\mathbf{a}| = \sqrt{(ca_1)^2 + (ca_2)^2 + (ca_3)^2} = |c||\mathbf{a}|.$$

The direction of the vector will reverse if  $c$  is negative, but otherwise is unaffected.

A vector where the sign is uncertain is called a **director**.

## ♣ Example

The electrostatic force on charged particle  $Q$  due to another charged particle  $q_1$  is

$$\mathbf{F} = K \frac{Qq_1}{r^2} \hat{\mathbf{r}} = K \frac{Qq_1}{r^3} \mathbf{r} \quad \text{where constant } K = \frac{1}{4\pi\epsilon_r\epsilon_0}$$

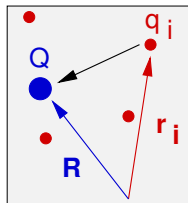
where  $\mathbf{r}$  is the vector from  $q_1$  to  $Q$ .

**Q:** Write down an expression for the force on  $Q$  at  $\mathbf{R}$  due to  $N$  charges  $q_i$  at  $\mathbf{r}_i$ ,  $i = 1, \dots, N$ .

**A:** The vector from  $q_i$  to  $Q$  is  $(\mathbf{R} - \mathbf{r}_i)$ .

The unit vector in that direction is  $(\mathbf{R} - \mathbf{r}_i)/|\mathbf{R} - \mathbf{r}_i|$ , so

$$\mathbf{F}(\mathbf{r}) = \sum_{i=1}^N K \frac{Qq_i}{|\mathbf{R} - \mathbf{r}_i|^3} (\mathbf{R} - \mathbf{r}_i)$$



Notice that we are thinking **algebraically** about **vectors** — not fussing about their components. Not a coordinate system in sight.

# Scalar or Dot product

The scalar product of two vectors results in a scalar quantity:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

Note that

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2 = a^2 .$$

These properties of the scalar product follow immediately:

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$   
(it commutes)
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$   
(it distributes w.r.t vector addition)
- $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda \mathbf{b})$   
(scalar multiple of a scalar product of two vectors)

# Geometrical interpretation of scalar product

Consider the square magnitude of the vector  $(\mathbf{a} - \mathbf{b})$ .

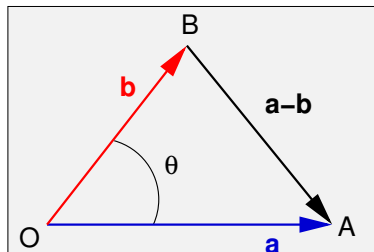
$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2(\mathbf{a} \cdot \mathbf{b}) \\ &= a^2 + b^2 - 2(\mathbf{a} \cdot \mathbf{b}) \end{aligned}$$

The *cosine rule* says length  $AB^2$  is  
 $|\mathbf{a} - \mathbf{b}|^2 = a^2 + b^2 - 2ab \cos \theta$

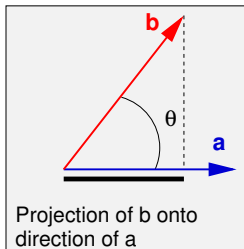
Hence

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta,$$

independent of the coord system.



# Projection of one vector onto the other



$\mathbf{b} \cdot \hat{\mathbf{a}} = b \cos \theta$  is the component of  $\mathbf{b}$  in the direction of  $\mathbf{a}$ .

$\mathbf{a} \cdot \hat{\mathbf{b}} = a \cos \theta$  is the component of  $\mathbf{a}$  in the direction of  $\mathbf{b}$ .

Projection is v. useful when the second vector is a unit vector.

- $\mathbf{a} \cdot \hat{\mathbf{i}}$  is the magnitude of the component of  $\mathbf{a}$  in the direction of  $\hat{\mathbf{i}}$ .
- $(\mathbf{a} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}}$  is the **vector component** of  $\mathbf{a}$  in the direction of  $\hat{\mathbf{i}}$ .
- To get the **vector component** of  $\mathbf{b}$  in the dirn of  $\mathbf{a}$

$$(\mathbf{b} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}} = \frac{1}{a^2}(\mathbf{b} \cdot \mathbf{a})\mathbf{a} .$$

# Orthonormal vectors and coordinates

In the particular case  $\mathbf{a} \cdot \mathbf{b} = 0$ , the angle between the two vectors is a right angle.

The vectors are said to be orthogonal — neither has a component in the direction of the other.

In 3D, an orthogonal coordinate system is characterised by

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

and

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$$

## A scalar product is an “inner product”

We have been writing vectors as **row vectors**  $\mathbf{a} = [a_1, a_2, a_3]$

It's convenient: it takes less space than writing column vectors

In matrix algebra, vectors are **column vectors**. So,  $\mathbf{M}\mathbf{a} = \mathbf{v}$  means

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

and row vectors are written as  $\mathbf{a}^\top$  ( $\mathbf{a}$  transpose).

Usually we can be relaxed — but care needed sometimes. Eg, the scalar product is also the **inner product** used in linear algebra.

The **inner product** is defined as  $\mathbf{a}^\top \mathbf{b}$

$$\mathbf{a}^\top \mathbf{b} = [a_1, a_2, a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a} \cdot \mathbf{b}$$



## ♣ Scalar Product Example 1

**Q:** Force  $\mathbf{F}$  is applied to an object as it moves by small amount  $\delta\mathbf{r}$ .  
What work is done on the object by the force?

**A:** The work done is equal to the component of force in the direction of the displacement multiplied by the displacement itself.  
This is just a scalar product:

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r} .$$

Of course you know that, but later we will see how to integrate such elements over particular paths as **line integrals**.

## ♣ Scalar Product Example 2

**Q:** A cube has four diagonals, connecting opposite vertices. What is the angle between an adjacent pair?

**A:**

First think coordinate-free ...

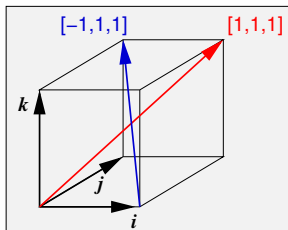
$$\theta = \cos^{-1} \left( \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{d_1 d_2} \right)$$

then detail ...

The directions of the diagonals are  $[\pm 1, \pm 1, \pm 1]$ . The ones shown in the figure are  $[1, 1, 1]$  and  $[-1, 1, 1]$ .

The angle is thus

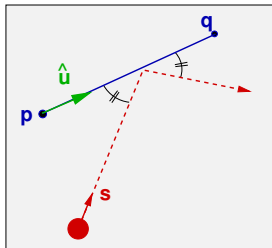
$$\begin{aligned} \theta &= \cos^{-1} \frac{[1, 1, 1] \cdot [-1, 1, 1]}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{-1^2 + 1^2 + 1^2}} \\ &= \cos^{-1}(1/3) \end{aligned}$$



## ♣ Scalar Product Example 3

**Q:** A pinball with velocity  $\mathbf{s}$  bounces (elastically) from a baffle whose endpoints are  $\mathbf{p}$  and  $\mathbf{q}$ . Find the velocity vector after bounce.

**A:**



Refer to coord frame with principal direction along the baffle

$$\hat{\mathbf{u}} = (\mathbf{q} - \mathbf{p}) / |\mathbf{q} - \mathbf{p}|$$

Physics tells us that velocities before and after are:

$$\mathbf{s}_{\text{before}} = \mathbf{s}_{\parallel} + \mathbf{s}_{\perp} \quad \mathbf{s}_{\text{after}} = \mathbf{s}_{\parallel} - \mathbf{s}_{\perp}$$

But

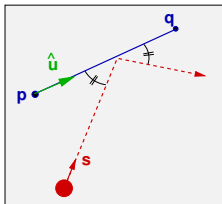
$$\mathbf{s}_{\parallel} = (\mathbf{s} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} \quad \Rightarrow \quad \mathbf{s}_{\perp} = \mathbf{s} - (\mathbf{s} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$$

So

$$\mathbf{s}_{\text{after}} = (\mathbf{s} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} - (\mathbf{s} - (\mathbf{s} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}) = 2(\mathbf{s} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} - \mathbf{s}$$

## ♣ Scalar Product Example 3

- Worth reflecting on this example ...
- Using vectors as complete entities (ie, not thinking about components) has made a tricky problem trivial to solve.
- Several languages (including Matlab) allow one to declare vector objects



```
p=[3;4];    q=[1;-1];    s=[1;2];
diff       = q-p;
uhat       = diff/norm(diff);
safter     = 2*dot(s,uhat)*uhat - s
```

- *You* think in vectors, while built in routines handle the detail of components
- ... Reflection over.

## Vector or cross product

The **vector product** of two vectors  $\mathbf{a}$  and  $\mathbf{b}$  is

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2) \hat{\mathbf{i}} + (a_3 b_1 - a_1 b_3) \hat{\mathbf{j}} + (a_1 b_2 - a_2 b_1) \hat{\mathbf{k}}.$$

Awkward to remember the above!

Instead use the **pseudo determinant**

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where the top row consists of vectors not scalars.

A determinant with two equal rows has value zero, so

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

It is also easily verified that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$$

so that  $\mathbf{a} \times \mathbf{b}$  is **orthogonal** to both  $\mathbf{a}$  and  $\mathbf{b}$ .

# Vector product

The magnitude of the vector product can be obtained by showing that

$$|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = a^2 b^2$$

from which it follows (independent of the coord system)

$$|\mathbf{a} \times \mathbf{b}| = ab \sin \theta ,$$

Proof? DIY

The vector product does not commute

It **anti-commutes**:  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ .

The vector product does not associate:

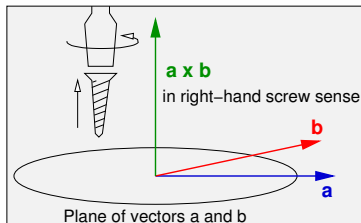
$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ .

# Vector Products

The vector product is orthogonal to both the vectors.

Need to specify the sense w.r.t these vectors.

Sense of the right handed screw



# Vector Products

Note that

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{\mathbf{k}} .$$

And in full:  $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$ ,  $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$ , and  $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$ .

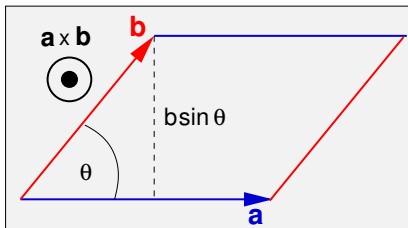
The cycle ordering is obvious.



# Geometrical interpretation of vector product

The magnitude of the vector product ( $\mathbf{a} \times \mathbf{b}$ ) is equal to the area of the parallelogram whose sides are parallel to, and have lengths equal to the magnitudes of, the vectors  $\mathbf{a}$  and  $\mathbf{b}$ .

Its direction is perpendicular to the parallelogram.



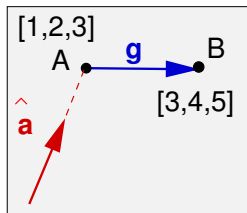
## ♣ Random question

**Q:**  $\mathbf{g}$  is vector from  $[1,2,3]$  to  $[3,4,5]$ .  
and  $\hat{\mathbf{a}}$  is the unit vector in dirn  $O \rightarrow A$ .

Find  $\hat{\mathbf{b}}$ , a UNIT vector along  $\mathbf{g} \times \hat{\mathbf{a}}$

Verify that  $\hat{\mathbf{b}}$  is perpendicular to  $\hat{\mathbf{a}}$ .

Find  $\hat{\mathbf{c}}$ , the third member of a r-h coord set  
 $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$ .



$$\mathbf{g} = [2, 2, 2].$$

$$\hat{\mathbf{a}} = [1, 2, 3]/\sqrt{14}$$

$$\mathbf{g} \times \hat{\mathbf{a}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = [2, -4, 2]$$

$$\Rightarrow \hat{\mathbf{b}} = [1, -2, 1]/\sqrt{6}$$

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = (1 \times 1 + 2 \times -2 + 1 \times 3) / \sqrt{84} = 0$$

$$\hat{\mathbf{c}} = \hat{\mathbf{a}} \times \hat{\mathbf{b}} = \frac{1}{\sqrt{6}\sqrt{14}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

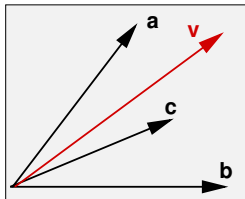
## ♣ Another random question

Q: If  $\hat{\mathbf{f}}$  and  $\hat{\mathbf{g}}$  are two unit vectors, what is the magnitude of the vector product  $(\hat{\mathbf{f}} \times \hat{\mathbf{g}})$ ?

## Components with a non-orthogonal basis set

Earlier we revised the idea of describing a vector as a linear sum of orthogonal components:  $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$ .

We also discussed components of a vector ... You'll recall  $v_1 = \mathbf{v} \cdot \hat{\mathbf{i}}$ , etc.



Suppose we replaced  $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$  with a **non-orthogonal, non-unit and non-coplanar** set of vectors  $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ .

By combining suitable amounts of these we can reach any point in 3-space, so any vector must be expressible as

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} .$$

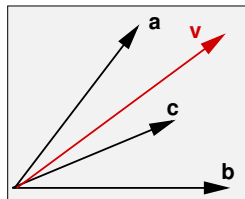
The question now is what are  $\alpha$ ,  $\beta$  and  $\gamma$ ?

# Components with a non-orthogonal basis set

You will be tempted to answer “the component of  $\mathbf{v}$  in the direction of  $\mathbf{a}$ ”.

That is,  $\alpha = (\mathbf{v} \cdot \mathbf{a})$

**THIS IS WRONG**



The way to find  $\alpha$  is to dot the RHS of the equation with a vector that is perpendicular to both  $\mathbf{b}$  and  $\mathbf{c}$ .

The obvious choice is  $(\mathbf{b} \times \mathbf{c})$ :

$$\begin{aligned} \mathbf{v} \cdot (\mathbf{b} \times \mathbf{c}) &= \alpha \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \beta \mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) + \gamma \mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}) \\ &= \alpha \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \end{aligned}$$

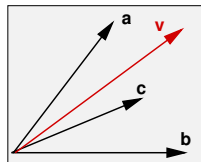
This is a scalar equation, so that

$$\alpha = \frac{\mathbf{v} \cdot (\mathbf{b} \times \mathbf{c})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})} \quad \text{and} \quad \beta = \frac{\mathbf{v} \cdot (\mathbf{c} \times \mathbf{a})}{\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})} \quad \text{and} \quad \gamma = \frac{\mathbf{v} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})}$$

## Components with a non-orthogonal basis set

$$\alpha = \frac{\mathbf{v} \cdot (\mathbf{b} \times \mathbf{c})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})} \quad \text{and} \quad \beta = \frac{\mathbf{v} \cdot (\mathbf{c} \times \mathbf{a})}{\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})} \quad \gamma = \frac{\mathbf{v} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})}$$

NB 1: that the results are unique. Though the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are not orthogonal, one cannot substitute, say, a bit more  $\mathbf{b}$  and  $\mathbf{c}$  for a loss of a bit of  $\mathbf{a}$ .



NB 2: The result specializes properly when the set is orthonormal. Eg ...

$$\alpha = \frac{\mathbf{v} \cdot (\hat{\mathbf{j}} \times \hat{\mathbf{k}})}{\hat{\mathbf{i}} \cdot (\hat{\mathbf{j}} \times \hat{\mathbf{k}})} = \frac{\mathbf{v} \cdot \hat{\mathbf{i}}}{\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}} = \mathbf{v} \cdot \hat{\mathbf{i}}.$$

NB 3: the expressions for  $\alpha$  etc fail when the denominator is zero. This occurs when the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are coplanar (later!).

# Summary

## We've revised and discussed ...

- Free, sliding and position vectors
- Coord frames and Vector components
- Equality, magnitude, Addition, Subtraction
- Scalar products, Vector Projection, Inner products
- Vector Products

## More routine stuff in Lecture 2 ...

Then the calculus starts

