Lecture 3

Differentiating Vector Functions of a Single Variable

It should be no great surprise that we often wish differentiate vector functions. For example, suppose you were driving along a wiggly road with position $\mathbf{r}(t)$ at time t. Differentiating $\mathbf{r}(t)$ wrt time should yield your velocity $\mathbf{v}(t)$, and differentiating $\mathbf{v}(t)$ should yield your acceleration. Let's see how to do this.

3.1 Differentiation of a vector

The derivative of a vector function $\mathbf{a}(p)$ of a single parameter p is

$$\frac{d\mathbf{a}}{dp} = \lim_{\delta p \to 0} \frac{\mathbf{a}(p + \delta p) - \mathbf{a}(p)}{\delta p} \ . \tag{3.1}$$

If we write $\mathbf{a}$ in terms of components relative to a FIXED coordinate system ($\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ constant)

$$\mathbf{a}(p) = a_1(p)\hat{\mathbf{i}} + a_2(p)\hat{\mathbf{j}} + a_3(p)\hat{\mathbf{k}} \tag{3.2}$$

then

$$\frac{d\mathbf{a}}{dp} = \frac{da_1}{dp}\hat{\mathbf{i}} + \frac{da_2}{dp}\hat{\mathbf{j}} + \frac{da_3}{dp}\hat{\mathbf{k}} \ . \tag{3.3}$$

That is, in order to differentiate a vector function, one simply differentiates each component separately.

For example, suppose $\mathbf{r}(t)$ is the position vector of an object moving w.r.t. the orgin.

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} \tag{3.4}$$

Then the instantaneous velocity is

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} \tag{3.5}$$

and the acceleration is

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \quad . \tag{3.6}$$

It also follows that all the familiar rules of differentiation apply, and they don't get altered by vector operations like scalar product and vector products. Thus, for example:

$$\frac{d}{dp}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dp} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dp} \qquad \frac{d}{dp}(\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dp} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dp} \quad . \tag{3.7}$$

Note that $d\mathbf{a}/dp$ has a different direction and a different magnitude from $\mathbf{a}$.

Likewise, as you might expect, the chain rule still applies. If $\mathbf{a} = \mathbf{a}(u)$ and $u = u(t)$, say:

$$\frac{d}{dt}\mathbf{a} = \frac{d\mathbf{a}}{du}\frac{du}{dt} \tag{3.8}$$

♣ Example #1

Q: The position of a vehicle is $\mathbf{r}(u)$ where u is the amount of fuel consumed by some time t. Work out an expression for the acceleration.

A: The velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{du}\frac{du}{dt} \tag{3.9}$$

$$\mathbf{a} = \frac{d}{dt}\frac{d\mathbf{r}}{dt} = \frac{d^2\mathbf{r}}{du^2}\left(\frac{du}{dt}\right)^2 + \frac{d\mathbf{r}}{du}\frac{d^2u}{dt^2} \tag{3.10}$$

♣ Example #2

Q: A 3D vector $\mathbf{a}$ of constant magnitude is varying over time. What can you say about the direction of $\dot{\mathbf{a}}$?

A: Using intuition: if only the direction is changing, then the vector must be tracing out points on the surface of a sphere. We would guess that the derivative $\dot{\mathbf{a}}$ is orthogonal to $\mathbf{a}$.

To prove this write

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = \mathbf{a} \cdot \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{a}}{dt} \cdot \mathbf{a} = 2\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} \quad . \tag{3.11}$$

But $(\mathbf{a} \cdot \mathbf{a}) = a^2$ which we are told is constant. So

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = 0 \qquad \Rightarrow 2\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0 \tag{3.12}$$

and hence $\mathbf{a}$ and $d\mathbf{a}/dt$ must be perpendicular.

3.2 Integration of a vector function

As with scalars, integration of a vector function of a single scalar variable is the reverse of differentiation. That is,

$$\int_{p_1}^{p_2} \left[\frac{d\mathbf{a}(p)}{dp} \right] dp = \mathbf{a}(p_2) - \mathbf{a}(p_1) \tag{3.13}$$

Eg from dynamics

$$\int_{t_1}^{t_2} \mathbf{a} \quad dt = \mathbf{v}(t_2) - \mathbf{v}(t_1) \tag{3.14}$$

However, other types of integral are possible, especially when the vector is a function of more than one variable. This requires the introduction of the concepts of scalar and vector fields. See lecture 4!

3.3 Space curves and derivatives

A "space curve" is simply a curve in 3D. We will assume that each point on the curve has a different position vector $\mathbf{r}$. Now suppose $\mathbf{r}$ is parameterized by p, so that by varying p we trace out the complete curve $\mathbf{r}(p)$.

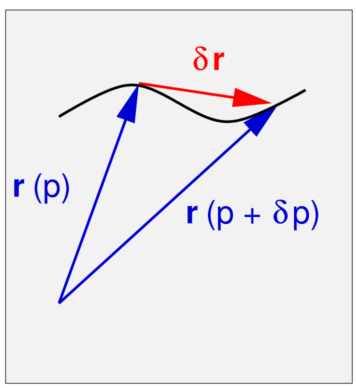


Figure 3.1: $\delta\mathbf{r}$ is a secant to the curve but, in the limit as $\delta p \to 0$, becomes a tangent.

Referring to Fig. 3.1, we can write

$$\mathbf{r}(p + \delta p) = \mathbf{r}(p) + \delta\mathbf{r} \ . \tag{3.15}$$

The small vector $\delta\mathbf{r}$ is obviously a secant to the curve, and $\delta\mathbf{r}/\delta p$ points in the same direction. It must do — we are just dividing a little vector by a little scalar.

In the limit as δp tends to zero

$$\lim_{\delta p \to 0} \frac{\delta\mathbf{r}}{\delta p} \to \frac{d\mathbf{r}}{dp} \tag{3.16}$$

a quantity which must be a tangent to the space curve. Note however that using a general parameter p there is nothing special about the magnitude of the tangent.

Fig. 3.2 shows just three of the infinity of ways of parametrizing the curve.

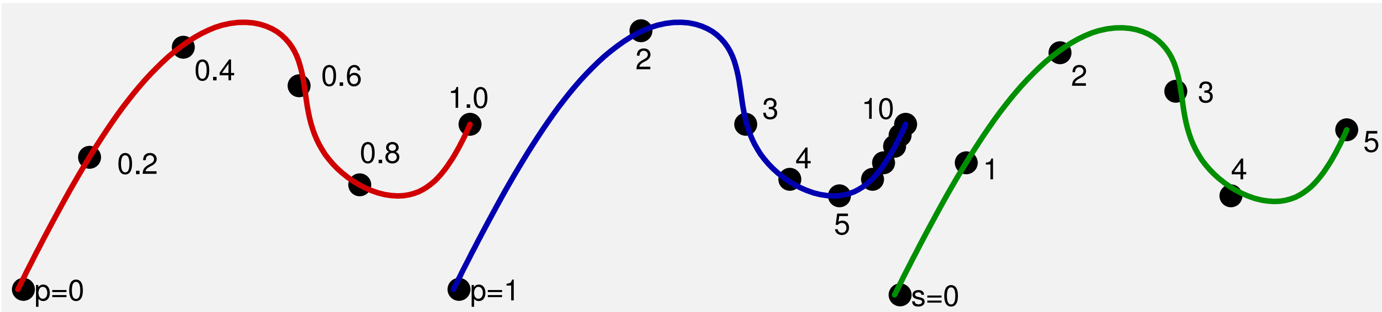


Figure 3.2: Different parametrizations describe the same curve. Arc-length s is special as it measures actual distance along the curve.

There is however one special parametrization, and that is when p measures arc-length. Usually denoted by s, the difference in arc-length s between two points on the curve is the actual distance travelled along the curve. But for infinitesimally small movements, $ds = |d\mathbf{r}|$, so that $d\mathbf{r}/ds$ must be of unit length.

We conclude that

> If a curve $\mathbf{r}(s)$ is parametrized by the arc length s
> $d\mathbf{r}/ds$ is everywhere a UNIT tangent to the curve.

More generally, however, p will not be arc-length. But the chain rule tells us that:

$$\frac{d\mathbf{r}}{dp} = \frac{d\mathbf{r}}{ds}\frac{ds}{dp} \tag{3.17}$$

So, the direction of the derivative is that of a tangent to the curve, and its magnitude is $|ds/dp|$, the rate of change of arc length w.r.t the parameter.

An interesting case is when p is time t

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} \tag{3.18}$$

So the vector velocity along the curve is the unit tangent times the scalar speed ds/dt.

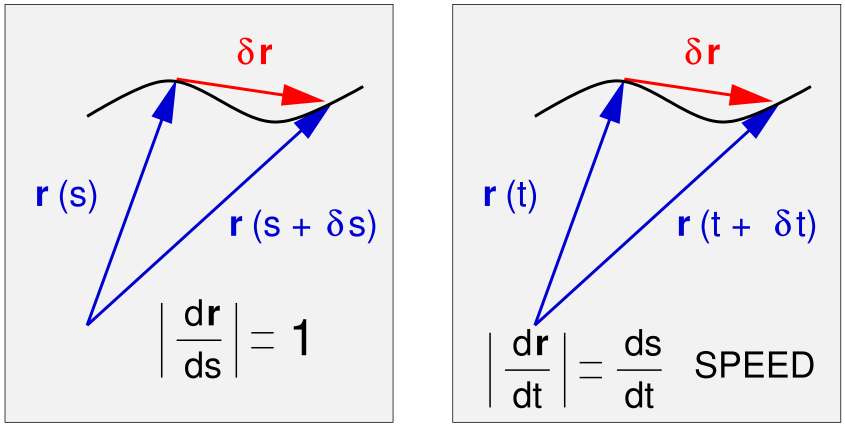


Figure 3.3:

♣ Example

Q: Draw the curve

$$\mathbf{r} = a\cos(\frac{s}{\sqrt{a^2+h^2}})\hat{\mathbf{\imath}} \quad + \quad a\sin(\frac{s}{\sqrt{a^2+h^2}})\hat{\mathbf{\jmath}} \quad + \quad \frac{hs}{\sqrt{a^2+h^2}}\hat{\mathbf{k}} \tag{3.19}$$

where s is arc length and h, a are constants. Show that the tangent $d\mathbf{r}/ds$ to the curve has a constant elevation angle w.r.t the xy-plane, and determine its magnitude.

A:

$$\frac{d\mathbf{r}}{ds} = -\frac{a}{\sqrt{a^2+h^2}}\sin()\,\hat{\mathbf{\imath}} + \frac{a}{\sqrt{a^2+h^2}}\cos()\,\hat{\mathbf{\jmath}} + \frac{h}{\sqrt{a^2+h^2}}\hat{\mathbf{k}} \tag{3.20}$$

The projection on the xy plane has magnitude $a/\sqrt{a^2+h^2}$ and in the z direction $h/\sqrt{a^2+h^2}$, so the elevation angle is a constant, $\tan^{-1}(h/a)$.

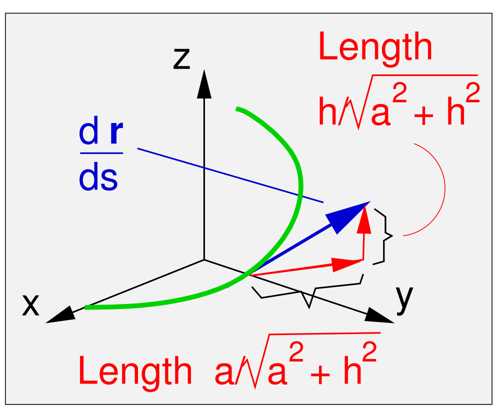


Figure 3.4:

We are expecting $d\mathbf{r}/ds = 1$, and indeed

$$\sqrt{a^2\sin^2() + a^2\cos^2() + h^2}/\sqrt{a^2+h^2} = 1. \tag{3.21}$$

The example used components, and it is worth stressing that the position vector $\mathbf{r}$ in Cartesian coordinates is

$$\mathbf{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}} \qquad \text{or using the parameter} \quad \mathbf{r}(p) = x(p)\hat{\mathbf{\imath}} + y(p)\hat{\mathbf{\jmath}} + z(p)\hat{\mathbf{k}} \; . \tag{3.22}$$

It follows that

$$d\mathbf{r} = dx\hat{\mathbf{\imath}} + dy\hat{\mathbf{\jmath}} + dz\hat{\mathbf{k}} \; . \tag{3.23}$$

But we have already noted that $ds = |d\mathbf{r}|$, hence it follows that

$$ds^2 = dx^2 + dy^2 + dz^2 \; . \tag{3.24}$$

This is akin to applying Pythagoras' theorem to a infinitesimal section of curve.

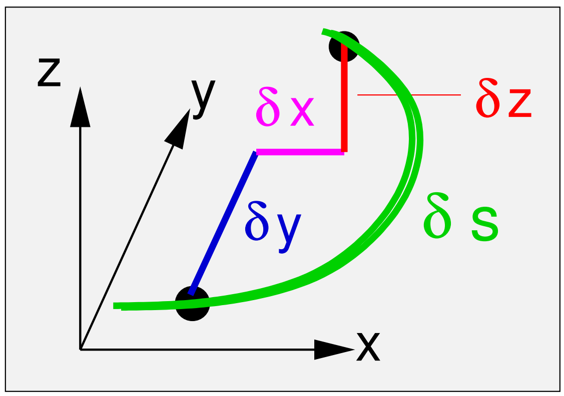


Figure 3.5:

So if a curve is parameterized in terms of p

$$\frac{ds}{dp} = \sqrt{\frac{dx}{dp}^2 + \frac{dy}{dp}^2 + \frac{dz}{dp}^2} \; , \tag{3.25}$$

which will be unity if and only if $p = s$.

If one can work out ds/dp one can easily find the relationship between s and p by integration. As an example, suppose in our earlier example we had parameterized the helix as $\mathbf{r} = a\cos p\hat{\mathbf{\imath}} + a\sin p\hat{\mathbf{\jmath}} + hp\hat{\mathbf{k}}$. Then

$$\begin{aligned} \frac{ds}{dp} &= \sqrt{\frac{dx}{dp}^2 + \frac{dy}{dp}^2 + \frac{dz}{dp}^2} \\ &= \sqrt{a^2\sin^2 p + a^2\cos^2 p + h^2} = \sqrt{a^2 + h^2} \end{aligned} \tag{3.26}$$

Integrating we see immediately that

$$p = s/\sqrt{a^2 + h^2} \; . \tag{3.27}$$

3.4 The Frenet-Serret relationships

We now know that we can specify points on a non-planar or space curve using distance or arc-length s along the wire.

We are now going to introduce a local orthogonal coordinate frame for each point s along the curve, ie one with its origin at $\mathbf{r}(s)$. To specify a coordinate frame we need three mutually perpendicular directions, and these should be *intrinsic* to the curve, not fixed in an external reference frame. The ideas were first suggested by two French mathematicians, F-J. Frenet and J. A. Serret.

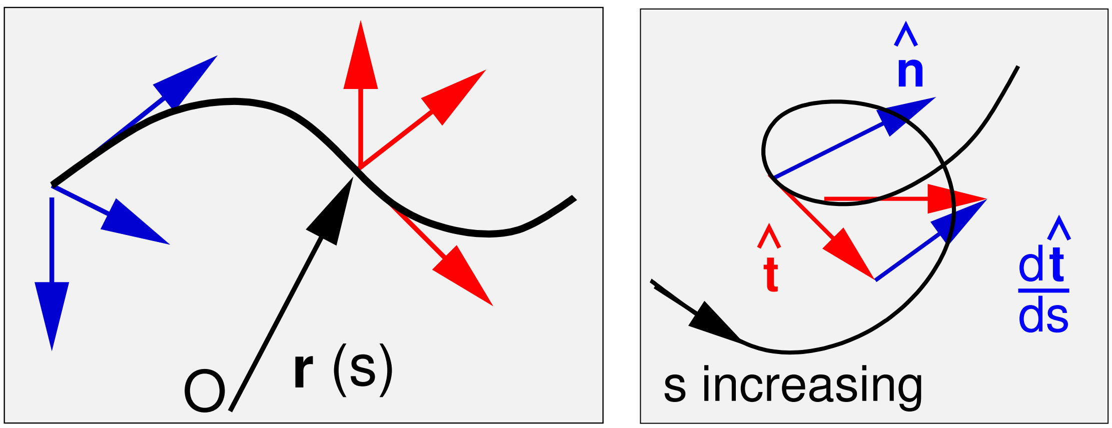


Figure 3.6:

1. **Tangent $\hat{\mathbf{t}}$**

   There is an obvious choice for the first direction at the point $\mathbf{r}(s)$, namely the **unit tangent** $\hat{\mathbf{t}}$. We already know that

   $$\hat{\mathbf{t}} = \frac{d\mathbf{r}(s)}{ds} \tag{3.28}$$

2. **Principal Normal $\hat{\mathbf{n}}$**

   Recall that earlier we proved that if $\mathbf{a}$ was a vector of constant magnitude that varies in direction over time then $d\mathbf{a}/dt$ was perpendicular to it. Because $\hat{\mathbf{t}}$ has constant magnitude but varies over s, $d\hat{\mathbf{t}}/ds$ must be perpendicular to $\hat{\mathbf{t}}$.

   Hence the principal normal $\hat{\mathbf{n}}$ is

   $$\frac{d\hat{\mathbf{t}}}{ds} = \kappa\hat{\mathbf{n}} : \text{ where } \kappa \geq 0 \ . \tag{3.29}$$

   κ is the **curvature**, and $\kappa = 0$ for a straight line. The plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ is called the **osculating plane**.

3. **The Binormal $\hat{\mathbf{b}}$**

The local coordinate frame is completed by defining the binormal

$$\hat{\mathbf{b}}(s) = \hat{\mathbf{t}}(s) \times \hat{\mathbf{n}}(s) \ . \tag{3.30}$$

Since $\hat{\mathbf{b}} \cdot \hat{\mathbf{t}} = 0$,

$$\frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{t}}}{ds} = \frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \kappa\hat{\mathbf{n}} = 0 \tag{3.31}$$

from which

$$\frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} = 0. \tag{3.32}$$

But this means that $d\hat{\mathbf{b}}/ds$ is along the direction of $\hat{\mathbf{n}}$, or

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau(s)\hat{\mathbf{n}}(s) \tag{3.33}$$

where τ is the **torsion**, and the negative sign is a matter of convention.

Differentiating $\hat{\mathbf{n}} \cdot \hat{\mathbf{t}} = 0$ and $\hat{\mathbf{n}} \cdot \hat{\mathbf{b}} = 0$, we find

$$\frac{d\hat{\mathbf{n}}}{ds} = -\kappa(s)\hat{\mathbf{t}}(s) + \tau(s)\hat{\mathbf{b}}(s). \tag{3.34}$$

We now have all three of

The Frenet-Serret relationships:

$$d\hat{\mathbf{t}}/ds = \kappa\hat{\mathbf{n}} \tag{3.35}$$

$$d\hat{\mathbf{n}}/ds = -\kappa(s)\hat{\mathbf{t}}(s) + \tau(s)\hat{\mathbf{b}}(s) \tag{3.36}$$

$$d\hat{\mathbf{b}}/ds = -\tau(s)\hat{\mathbf{n}}(s) \tag{3.37}$$

♣ **Example**

Q Derive $\kappa(s)$ and $\tau(s)$ for the helix

$$\mathbf{r}(s) = a\cos\left(\frac{s}{\beta}\right)\hat{\mathbf{\imath}} + a\sin\left(\frac{s}{\beta}\right)\hat{\mathbf{\jmath}} + h\left(\frac{s}{\beta}\right)\hat{\mathbf{k}}; \qquad \beta = \sqrt{a^2 + h^2} \tag{3.38}$$

and comment on their values.

A We found the unit tangent earlier as

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \left[-\frac{a}{\beta}\sin\left(\frac{s}{\beta}\right), \ \frac{a}{\beta}\cos\left(\frac{s}{\beta}\right), \ \frac{h}{\beta}\right] \ . \tag{3.39}$$

Differentiation gives

$$\kappa\hat{\mathbf{n}} = \frac{d\hat{\mathbf{t}}}{ds} = \left[-\frac{a}{\beta^2}\cos\left(\frac{s}{\beta}\right), \quad -\frac{a}{\beta^2}\sin\left(\frac{s}{\beta}\right), \quad 0\right] \tag{3.40}$$

Curvature is always positive, so

$$\kappa = \frac{a}{\beta^2} \qquad \hat{\mathbf{n}} = \left[-\cos\left(\frac{s}{\beta}\right), \quad -\sin\left(\frac{s}{\beta}\right), \quad 0\right] . \tag{3.41}$$

So the curvature is constant, and the normal is parallel to the xy-plane.

Now use

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ (-a/\beta)S & (a/\beta)C & (h/\beta) \\ -C & -S & 0 \end{vmatrix} = \left[\frac{h}{\beta}\sin\left(\frac{s}{\beta}\right), \quad -\frac{h}{\beta}\cos\left(\frac{s}{\beta}\right), \quad \frac{a}{\beta}\right] \tag{3.42}$$

and differentiate $\hat{\mathbf{b}}$ to find an expression for the torsion

$$\frac{d\hat{\mathbf{b}}}{ds} = \left[\frac{h}{\beta^2}\cos\left(\frac{s}{\beta}\right), \quad \frac{h}{\beta^2}\sin\left(\frac{s}{\beta}\right), \quad 0\right] = \frac{-h}{\beta^2}\hat{\mathbf{n}} \tag{3.43}$$

so the torsion is

$$\tau = \frac{h}{\beta^2} \tag{3.44}$$

again a constant.

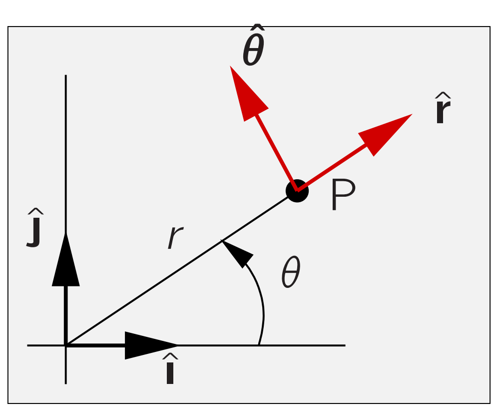
3.5 Derivative (eg velocity) components in plane polars

In plane polar coordinates, the radius vector of any point P is given by

$$\begin{aligned} \mathbf{r} &= r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}} && (3.45) \\ &= r\hat{\mathbf{r}} && (3.46) \end{aligned}$$

where we have introduced the unit radial vector

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}} \quad . \tag{3.47}$$



The other "natural" (we'll see why in a later lecture) unit vector in plane polars is orthogonal to $\hat{\mathbf{r}}$ and is

$$\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}} \tag{3.48}$$

so that $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = 1$ and $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$.

Now suppose P is moving so that $\mathbf{r}$ is a function of time t. Its velocity is

$$\begin{aligned}
\dot{\mathbf{r}} = \frac{d}{dt}(r\hat{\mathbf{r}}) &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\
&= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}) \\
&= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \\
&= \text{radial} + \text{tangential}
\end{aligned} \tag{3.49}$$

The radial and tangential components of velocity of P are therefore dr/dt and $rd\theta/dt$, respectively.

Differentiating a second time gives the acceleration of P

$$\begin{aligned}
\ddot{\mathbf{r}} &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} - r\frac{d\theta}{dt}\frac{d\theta}{dt}\hat{\mathbf{r}} \\
&= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{r}} + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{\boldsymbol{\theta}}
\end{aligned} \tag{3.50}$$

Three obvious cases are:

$$\theta \text{ const:} \quad \ddot{\mathbf{r}} = \frac{d^2r}{dt^2}\hat{\mathbf{r}} \tag{3.51}$$

$$r \text{ const:} \quad \ddot{\mathbf{r}} = -r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}}$$

$$r \text{ and } d\theta/dt \text{ const:} \quad \ddot{\mathbf{r}} = -r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}} \tag{3.52}$$

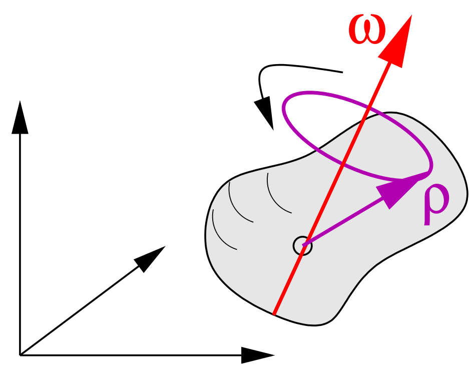
3.6 Rotating systems

Consider a body which is rotating with constant angular velocity $\boldsymbol{\omega}$ about some axis passing through the origin. Assume the origin is fixed, and that we are sitting in a fixed coordinate system $Oxyz$.

If $\boldsymbol{\rho}$ is a vector of constant magnitude and constant direction in the rotating system, then it must be a function of t in the fixed system.

At any instant as observed in the fixed system

$$\frac{d\boldsymbol{\rho}}{dt} = \boldsymbol{\omega} \times \boldsymbol{\rho}(t). \tag{3.53}$$



Note that: $d\boldsymbol{\rho}/dt$
– will have fixed magnitude,
– will always be perpendicular to the axis of rotation
– will vary in direction within those constraints.
The point $\boldsymbol{\rho}(t)$ will move in a plane in the fixed system.

3.6.1 Rotation: Part 2

Now consider a set of mutually orthogonal unit vectors $\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}}$ *attached to the rotating system*. In the fixed frame, each of $\hat{\mathbf{l}}, \hat{\mathbf{m}}$, and $\hat{\mathbf{n}}$ has a time dependence:

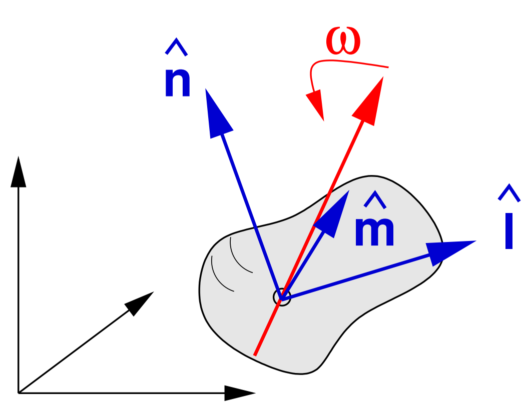
$$\frac{d\hat{\mathbf{l}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{l}} \qquad \frac{d\hat{\mathbf{m}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{m}} \qquad \frac{d\hat{\mathbf{n}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{n}} \tag{3.54}$$

Note (1) that the angular velocity vector $\boldsymbol{\omega}$ points in the same direction as the axis of rotation, and is fixed both with respect to the rotating frame *and* the fixed frame.

Note (2) that since each of $d\hat{\mathbf{l}}/dt, d\hat{\mathbf{m}}/dt, d\hat{\mathbf{n}}/dt$ is perpendicular to $\boldsymbol{\omega}$ they must be coplanar.

Let $\boldsymbol{\rho} = \rho_1\hat{\mathbf{l}} + \rho_2\hat{\mathbf{m}} + \rho_3\hat{\mathbf{n}}$ be a constant vector in the rotating frame, so that $\rho_{1,2,3}$ are constant. Its rate of change in fixed frame is

$$\begin{aligned}\frac{d\boldsymbol{\rho}}{dt} &= \boldsymbol{\omega} \times \boldsymbol{\rho} \qquad (3.55)\\ &= \rho_1(\boldsymbol{\omega} \times \hat{\mathbf{l}}) + \rho_2(\boldsymbol{\omega} \times \hat{\mathbf{m}}) + \rho_3(\boldsymbol{\omega} \times \hat{\mathbf{n}})\\ &= \rho_1\frac{d\hat{\mathbf{l}}}{dt} + \rho_2\frac{d\hat{\mathbf{m}}}{dt} + \rho_3\frac{d\hat{\mathbf{n}}}{dt}\end{aligned}$$



So, as expected, its time dependence derives from the time dependence of $\hat{\mathbf{l}}(t)$,$\hat{\mathbf{m}}$(t), $\hat{\mathbf{n}}(t)$, and not from its coefficients with respect to this basis set, which were constants.

3.6.2 Rotation: Part 3

Now suppose $\boldsymbol{\rho}$ is the position vector of a point P which **moves** in the rotating frame. It will have two contributions to motion with respect to the fixed frame, one due to it motion within the rotating frame, and one due to the rotation itself.

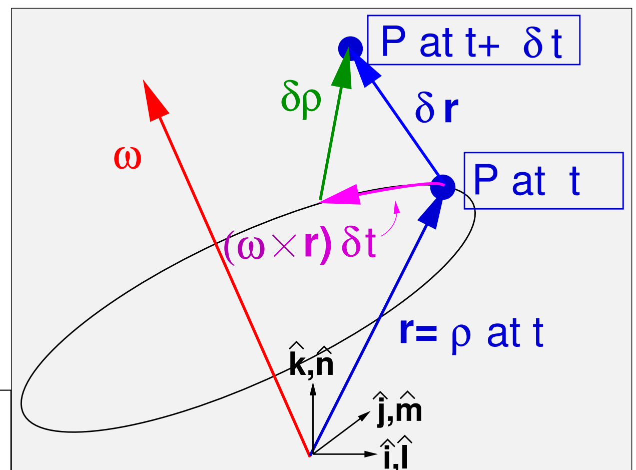
Let $\boldsymbol{\rho}$ be defined in a rotating coordinate frame which is, **instantaneously**, aligned with the fixed coord system. So at time t, **and only at time** t, $\mathbf{r} = \boldsymbol{\rho}$.

Over a period δt, at time t, the component of the motion in the fixed frame due to rotation is

$$(\boldsymbol{\omega} \times \boldsymbol{\rho})\delta t = (\boldsymbol{\omega} \times \mathbf{r})\delta t \ . \tag{3.56}$$

If the component of "independent" motion in the rotating frame is $\delta\boldsymbol{\rho}$, then the overall movement in time δt is

$$\delta \mathbf{r} \ = \ \delta\boldsymbol{\rho} \ + \ (\boldsymbol{\omega} \times \mathbf{r})\delta t \ . \tag{3.57}$$



So the **instantaneous velocity** in the fixed frame is

$$\frac{d\mathbf{r}}{dt} \ = \ \frac{D\boldsymbol{\rho}}{Dt} \ + \ \boldsymbol{\omega} \times \mathbf{r} \tag{3.58}$$

NB! The capital D's are used to indicate differentiation in the rotating frame.

3.6.3 Rotation 4: Instantaneous acceleration

Our previous result is a general one relating the time derivatives of any vector in rotating and non-rotating frames. Because the frames are instantaneously aligned at t, any vector in the fixed frame has the same value in the rotating frame — just as $\mathbf{r} = \boldsymbol{\rho}$.

So, using operator notation,

$$\ddot{\mathbf{r}} = \left[\frac{D}{Dt} + \boldsymbol{\omega}\times\right]\dot{\mathbf{r}} = \left[\frac{D}{Dt} + \boldsymbol{\omega}\times\right]\left(\frac{D\boldsymbol{\rho}}{Dt} + \boldsymbol{\omega} \times \mathbf{r}\right) \tag{3.59}$$

The **instantaneous acceleration** is therefore

$$\ddot{\mathbf{r}} = \frac{D^2\boldsymbol{\rho}}{Dt^2} + 2\boldsymbol{\omega} \times \frac{D\boldsymbol{\rho}}{Dt} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{3.60}$$

- The first term is the acceleration of the point P in the rotating frame measured in the rotating frame.

- The last term is the centripetal acceleration to due to the rotation. (Yes! Its magnitude is $\omega^2 r$ and its direction is that of $-\mathbf{r}$. Check it out.)
- The middle term is an extra term which arises because of the velocity of P in the rotating frame. It is known as the **Coriolis acceleration**, named after the French engineer who first identified it.

Because of the rotation of the earth, the Coriolis acceleration is of great importance in meteorology and accounts for the occurrence of high pressure anti-cyclones and low pressure cyclones in the northern hemisphere, in which the Coriolis acceleration is produced by a pressure gradient. It is also a very important component of the acceleration (hence the force exerted) by a rapidly moving robot arm, whose links whirl rapidly about rotary joints.

♣ Example

Q: Find the instantaneous acceleration of a projectile fired along a line of longitude (with angular velocity of $\boldsymbol{\gamma}$ constant relative to the sphere) if the sphere is rotating with angular velocity $\boldsymbol{\omega}$.

A: In the rotating frame

$$\frac{D\boldsymbol{\rho}}{Dt} = \boldsymbol{\gamma} \times \boldsymbol{\rho} \qquad \text{and} \qquad \frac{D^2\boldsymbol{\rho}}{Dt^2} = \boldsymbol{\gamma} \times \frac{D\boldsymbol{\rho}}{Dt} = \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \boldsymbol{\rho}) \tag{3.61}$$

So the in the fixed reference frame

$$\ddot{\mathbf{r}} = \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \mathbf{r}) + 2\boldsymbol{\omega} \times (\boldsymbol{\gamma} \times \mathbf{r}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \; . \tag{3.62}$$

The first term is the centripetal acceleration due to the projectile moving around the sphere — which it does because of the gravitational force. The last term is the centripetal acceleration resulting from the rotation of the sphere. The middle term is the Coriolis acceleration.

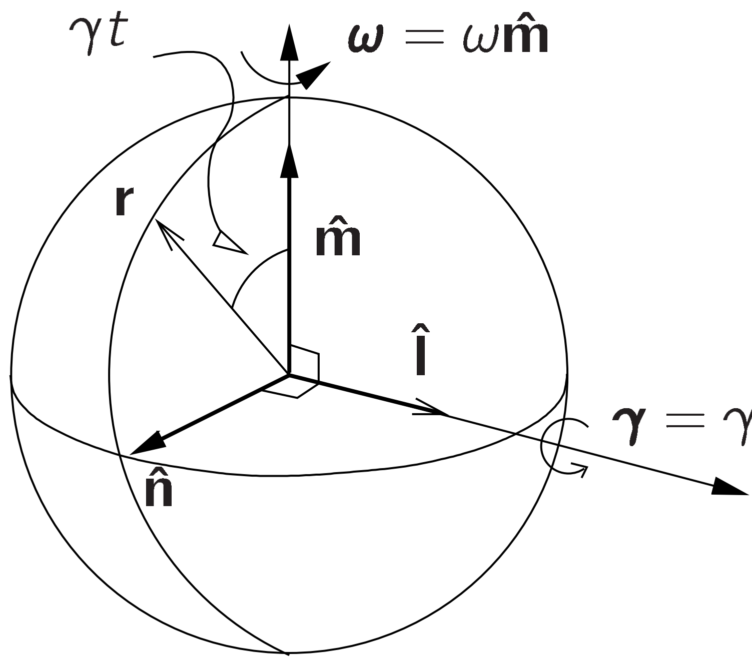


Figure 3.7: Coriolis example.

Using Fig. 3.7, at some instant t let

$$\mathbf{r} = \boldsymbol{\rho}(t) = R\cos(\gamma t)\hat{\mathbf{m}} \quad + \quad R\sin(\gamma t)\hat{\mathbf{n}} \tag{3.63}$$

Then

$$\boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \mathbf{r}) = (\boldsymbol{\gamma} \cdot \mathbf{r})\boldsymbol{\gamma} - \gamma^2\mathbf{r} = -\gamma^2\mathbf{r} \ , \tag{3.64}$$

as $\boldsymbol{\gamma} = \gamma\hat{\mathbf{l}}$. Check the direction — the negative sign means it points *towards* the centre of the sphere, which is as expected. Likewise the last term can be obtained as

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\omega^2 R\sin(\gamma t)\hat{\mathbf{n}} \tag{3.65}$$

Note that it is perpendicular to the axis of rotation $\hat{\mathbf{m}}$, and because of the minus sign, directed towards the axis)

The Coriolis term is derived as:

$$2\boldsymbol{\omega} \times \frac{D\boldsymbol{\rho}}{Dt} = 2\begin{bmatrix}0\\ \omega\\ 0\end{bmatrix} \times \left(\begin{bmatrix}\gamma\\ 0\\ 0\end{bmatrix} \times \begin{bmatrix}0\\ R\cos\gamma t\\ R\sin\gamma t\end{bmatrix}\right) \tag{3.66}$$

$$= 2\omega\gamma R\cos\gamma t\hat{\mathbf{l}} \tag{3.67}$$

Now consider a rocket on rails which stretch north from the equator. As the rocket travels north it experiences the Coriolis force (exerted by the rails):

$$\begin{array}{cccc} 2 & \gamma & \omega & R\cos\gamma t \;\; \hat{\mathbf{I}} \\ \text{+ve} & \text{-ve} & \text{+ve} & \text{+ve} \end{array}$$

Hence the coriolis force is in the direction opposed to $\hat{\mathbf{I}}$ (i.e. in the opposite direction to the earth's rotation). In the absence of the rails (or atmosphere) the rocket's tangetial speed (relative to the surface of the earth) is *greater* than the speed of the surface of the earth underneath it (since the radius of successive lines of latitude decreases) so it would (to an observer on the earth) appear to deflect to the east. The rails provide a coriolis force keeping it on the same meridian.

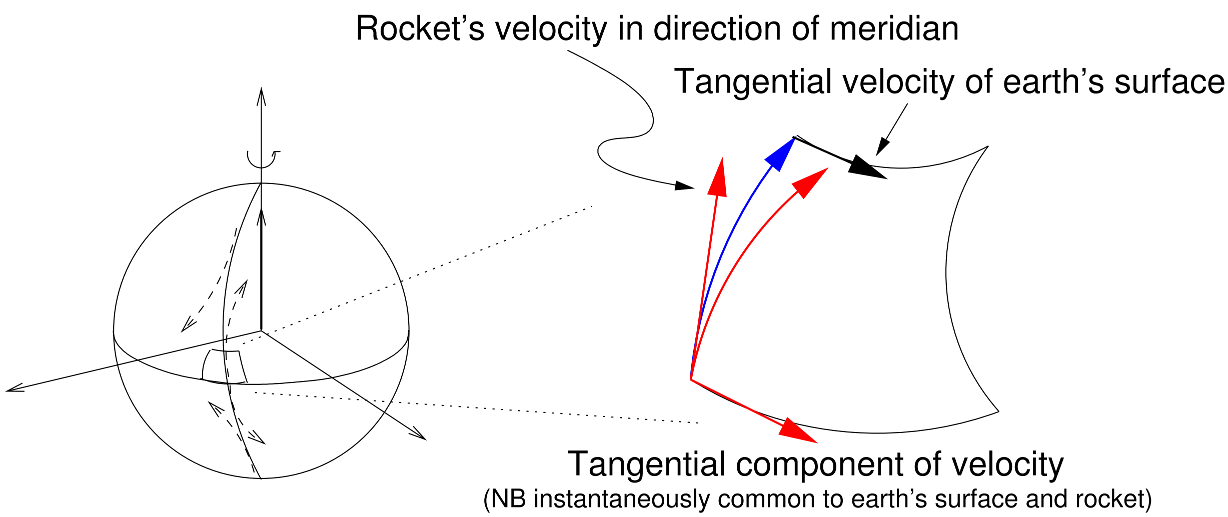
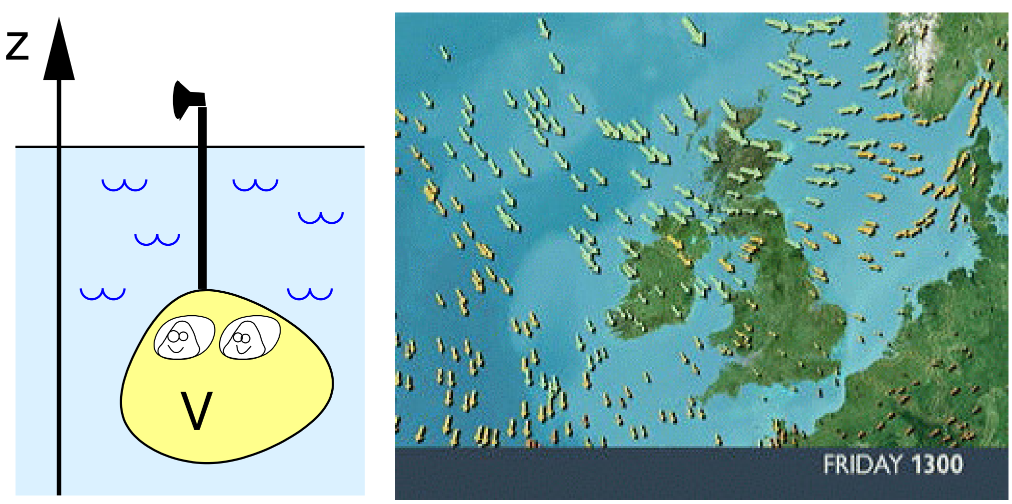


Figure 3.8: Rocket example

Lecture 4

Line, Surface and Volume Integrals. Curvilinear coordinates.

In much of the rest of the course, we will be concerned not with individual vectors, but with scalars and vectors which are defined over regions in space — scalar and vector *fields*. When a scalar function $u(\mathbf{r})$ is determined or defined at each position $\mathbf{r}$ in some region, we say that u is a **scalar field** in that region. Similarly, if a vector function $\mathbf{v}(\mathbf{r})$ is defined at each point, then $\mathbf{v}$ is a **vector field** in that region. As you will see, in **field theory** our aim is to derive statements about the bulk properties of scalar and vector fields, rather than to deal with individual scalars or vectors. Familiar examples of each are shown in Fig. 4.1.



(a) (b)

Figure 4.1: Examples of (a) a scalar field (pressure); (b) a vector field (wind velocity)

In this lecture we introduce line, surface and volume integrals, and consider how these are defined in non-Cartesian, *curvilinear coordinates*.

4.1 Line integrals through fields

Line integrals are concerned with measuring the integrated interaction with a field as you move through it on some defined path. Eg, given a map showing the pollution density field in Oxford, you may wish to work out how much pollution you breath in when cycling from college to the Department via different routes.

First recall the definition of an integral for a scalar function $f(x)$ of a single scalar variable x. One assumes a set of n samples $f_i = f(x_i)$ spaced by δx_i. One forms the limit of the sum of the products $f(x_i)\delta x_i$ as the number of samples tends to infinity

$$\int f(x)dx = \lim_{\substack{n \to \infty \\ \delta x_i \to 0}} \sum_{i=1}^{n} f_i \delta x_i \quad . \tag{4.1}$$

For a smooth function, it is irrelevant how the function is subdivided.

In a vector line integral, the path (a space curve!) L along which the integral is to be evaluated is split into a large number of *vector* segments $\delta\mathbf{r}_i$. Each line segment is then multiplied by the quantity associated with that point in space, the products are then summed and the limit taken as the lengths of the segments tend to zero.

There are three types of integral we have to think about, depending on the nature of the product:

1. Integrand $U(\mathbf{r})$ is a scalar field, hence the integral is a vector.

$$\mathbf{I} = \int_L U(\mathbf{r})d\mathbf{r} \qquad \left(= \lim_{\delta\mathbf{r}_i \to 0} \sum_i U_i \delta\mathbf{r}_i . \right) \tag{4.2}$$

2. Integrand $\mathbf{a}(\mathbf{r})$ is a vector field dotted with $d\mathbf{r}$ hence the integral is a scalar:

$$I = \int_L \mathbf{a}(\mathbf{r}) \cdot d\mathbf{r} \qquad \left(= \lim_{\delta\mathbf{r}_i \to 0} \sum_i \mathbf{a}_i \cdot \delta\mathbf{r}_i . \right) \tag{4.3}$$

3. Integrand $\mathbf{a}(\mathbf{r})$ is a vector field crossed with $d\mathbf{r}$ hence vector result.

$$\mathbf{I} = \int_L \mathbf{a}(\mathbf{r}) \times d\mathbf{r} \qquad \left(= \lim_{\delta\mathbf{r}_i \to 0} \sum_i \mathbf{a}_i \times \delta\mathbf{r}_i . \right) \tag{4.4}$$

Note immediately that unlike an integral in a single scalar variable, there are many paths L from start point $\mathbf{r}_A$ to end point $\mathbf{r}_B$, **and in general the integral will depend on the path taken**.

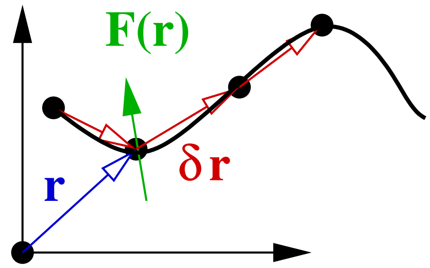


Figure 4.2: Line integral. In the diagram $\mathbf{F}(\mathbf{r})$ is a vector field, but it could be replace with scalar field $U(\mathbf{r})$.

4.1.1 Physical examples of line integrals

i) The total work done by a force $\mathbf{F}$ as it moves a point from A to B along a given path C is given by a line integral of type 2 above. If the force acts at point $\mathbf{r}$ and the instantaneous displacement along curve C is $d\mathbf{r}$ then the infinitesimal work done is $dW = \mathbf{F} \cdot d\mathbf{r}$, and so the total work done traversing the path is

$$W_C = \int_C \mathbf{F} \cdot d\mathbf{r} \tag{4.5}$$

ii) Ampère's law relating magnetic intensity $\mathbf{H}$ to linked current can be written as

$$\oint_C \mathbf{H} \cdot d\mathbf{r} = I \tag{4.6}$$

where I is the current enclosed by the closed path C.

iii) The force on an element of wire carrying current I, placed in a magnetic field of strength $\mathbf{B}$, is $d\mathbf{F} = I d\mathbf{r} \times \mathbf{B}$. So if a loop C of this wire is placed in the field, the total force will be and integral of type 3 above:

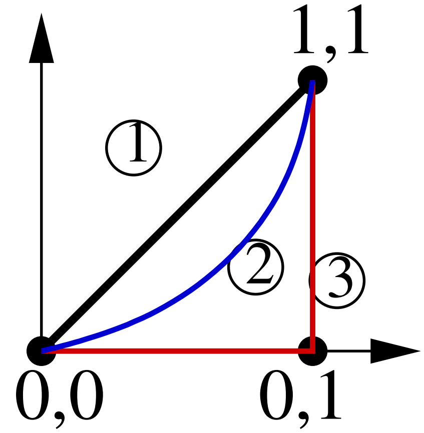
$$\mathbf{F} = I \oint_C d\mathbf{r} \times \mathbf{B} \tag{4.7}$$

Note that the expressions above are beautifully compact in vector notation, and are all independent of coordinate system. Of course when evaluating them we need to choose a coordinate system: often this is the standard Cartesian coordinate system (as in the worked examples below), but need not be.

♣ Examples

Q1: An example in the xy-plane. A force $\mathbf{F} = x^2y\hat{\mathbf{\imath}} + xy^2\hat{\mathbf{\jmath}}$ acts on a body at it moves between $(0,0)$ and $(1,1)$. Determine the work done when the path is

1. along the line $y = x$.
2. along the curve $y = x^n$.
3. along the x axis to the point $(1,0)$ and then along the line $x = 1$.



A1: This is an example of the "type 2" line integral. In plane Cartesians, $d\mathbf{r} = \hat{\mathbf{\imath}}dx + \hat{\mathbf{\jmath}}dy$. Then the work done is

$$\int_L \mathbf{F}\cdot d\mathbf{r} = \int_L (x^2y\,dx + xy^2dy)\ . \tag{4.8}$$

1. For the path $y = x$ we find that $dy = dx$. So it is easiest to convert all y references to x.

$$\int_{(0,0)}^{(1,1)}(x^2y\,dx + xy^2dy) = \int_{x=0}^{x=1}(x^2x\,dx + xx^2dx) = \int_{x=0}^{x=1}2x^3dx = \left[x^4/2\right|_{x=0}^{x=1} = 1/2\ . \tag{4.9}$$

2. For the path $y = x^n$ we find that $dy = nx^{n-1}dx$, so again it is easiest to convert all y references to x.

$$\int_{(0,0)}^{(1,1)}(x^2y\,dx + xy^2dy) = \int_{x=0}^{x=1}(x^{n+2}dx + nx^{n-1}.x.x^{2n}dx) \tag{4.10}$$

$$= \int_{x=0}^{x=1}(x^{n+2}dx + nx^{3n}dx) \tag{4.11}$$

$$= \frac{1}{n+3} + \frac{n}{3n+1} \tag{4.12}$$

3. This path is not smooth, so break it into two. Along the first section, $y = 0$ and $dy = 0$, and on the second $x = 1$ and $dx = 0$, so

$$\int_A^B (x^2y\,dx + xy^2dy) = \int_{x=0}^{x=1}(x^2 0\,dx) + \int_{y=0}^{y=1}1.y^2dy = 0 + \left[y^3/3\right|_{y=0}^{y=1} = 1/3\ . \tag{4.13}$$

So in general the integral depends on the path taken. Notice that answer (1) is the same as answer (2) when $n = 1$, and that answer (3) is the limiting value of answer (2) as $n \to \infty$.

Q2: Repeat part (2) using the Force $\mathbf{F} = xy^2\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}}$.

A2: For the path $y = x^n$ we find that $dy = nx^{n-1}dx$, so

$$\int_{(0,0)}^{(1,1)} (y^2xdx + yx^2dy) = \int_{x=0}^{x=1} (x^{2n+1}dx + nx^{n-1}.x^2.x^n dx) \tag{4.14}$$

$$= \int_{x=0}^{x=1} (x^{2n+1}dx + nx^{2n+1}dx) \tag{4.15}$$

$$= \frac{1}{2n+2} + \frac{n}{2n+2} \tag{4.16}$$

$$= \frac{1}{2} \quad \text{independent of } n \tag{4.17}$$

4.1.2 Line integrals in Conservative fields

In example #2, the line integral has the same value for the whole range of paths. We now prove that it is wholly independant of path.

Consider the function $g(x, y) = x^2y^2/2$. Using the definition of the perfect or total differential

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy \quad \text{and in this case} \quad dg = y^2xdx + yx^2dy \ . \tag{4.18}$$

So our line is actually

$$\int_A^B (y^2xdx + yx^2dy) = \int_A^B dg \quad = g_B - g_A \ . \tag{4.19}$$

This depends solely on the value of g at the start and end points, and not at all on the path used to get from A to B.

Such a vector field is called **conservative**.

If $\mathbf{F}$ is a conservative field, the line integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

An immediate corollary is that

If $\mathbf{F}$ is a convervative field, the line integral around a closed path $\oint \mathbf{F} \cdot d\mathbf{r}$ is zero.

There will be more to say on this when we consider the gradient operator.

♣ Example

Q: In an eletric field **E**, the potential function is $\phi = -\int \mathbf{E} \cdot d\mathbf{r}$. State whether or not **E** is a conservative field.

A: We know that the the potential difference around any loop is zero. This must mean that **E** is conservative.

4.1.3 A note on line integrals defined in terms of arc length

(You might wish to return to this later when you are more confident with the rest of the material.)

Line integrals are often defined in terms of scalar arc length. They don't appear to involve vectors, but actually they are another form of type 2 defined earlier.

The integrals usually appears as follows

$$I = \int_L F(x, y, z) ds \tag{4.20}$$

and most often the path L is along a curve defined parametrically as $x = x(p)$, $y = y(p)$, $z = z(p)$ where p is some parameter. Convert the function to $F(p)$, writing

$$I = \int_{p_{\text{start}}}^{p_{\text{end}}} F(p) \frac{ds}{dp}\, dp \tag{4.21}$$

where

$$\frac{ds}{dp} = \left[\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2 \right]^{1/2} . \tag{4.22}$$

Note that the parameter p could be arc-length s itself, in which case $ds/dp = 1$ of course! Another possibility is that the parameter p is x — that is we are told $y = y(x)$ and $z = z(x)$. Then

$$I = \int_{x_{\text{start}}}^{x_{\text{end}}} F(x) \left[1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2 \right]^{1/2} dx . \tag{4.23}$$

4.2 Surface integrals

The surface S over which the integral is to be evaluated is now divided into infinitesimal vector elements of area $d\mathbf{S}$, the direction of the vector $d\mathbf{S}$ representing the direction of the surface normal and its magnitude representing the area of the element.

Again there are three possibilities:

- $\int_S U d\mathbf{S}$ — scalar field U; vector integral.
- $\int_S \mathbf{a} \cdot d\mathbf{S}$ — vector field $\mathbf{a}$; scalar integral.
- $\int_S \mathbf{a} \times d\mathbf{S}$ — vector field $\mathbf{a}$; vector integral.

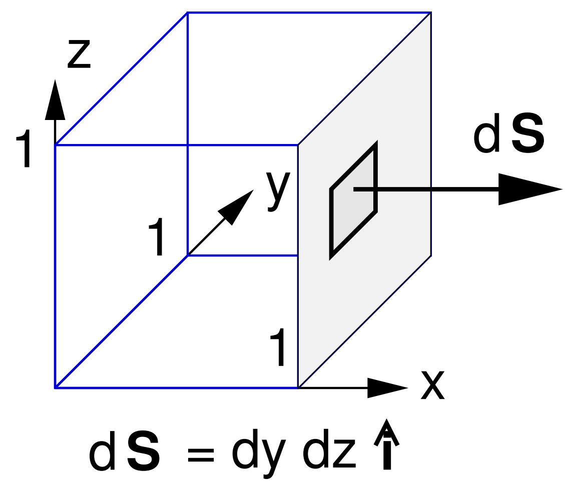
Physical examples of surface integrals with vectors often involve the idea of *flux* of a vector field through a surface, $\int_S \mathbf{a} \cdot d\mathbf{S}$. For example the mass of fluid crossing a surface S in time dt is $dM = \rho \mathbf{v} \cdot d\mathbf{S} dt$ where $\rho(\mathbf{r})$ is the fluid density and $\mathbf{v}(\mathbf{r})$ is the fluid velocity. The total mass flux can be expressed as a surface integral:

$$\Phi_M = \int_S \rho(\mathbf{r})\mathbf{v}(\mathbf{r}) \cdot d\mathbf{S} \tag{4.24}$$

Again, note that this expression is coordinate free. Our first example below using Cartesians, but — as with line integrals — symmetry may lead us to a different more natural coordinate system.

♣ Example

Q: Evaluate $\int \mathbf{F} \cdot d\mathbf{S}$ over the $x = 1$ side of the cube shown in the figure when $\mathbf{F} = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}$.
A: $d\mathbf{S}$ is perpendicular to the surface. Its $\pm$ direction actually depends on the nature of the problem. More often than not, the surface will enclose a volume, and the surface direction is taken as everywhere emanating from the interior.



Hence for the face of the cube at $x = 1$

$$d\mathbf{S} = dydz\hat{\mathbf{i}} \tag{4.25}$$

and

$$\int \mathbf{F} \cdot d\mathbf{S} = \int_{z=0}^{1} \int_{y=0}^{1} y dy dz = \frac{1}{2} y^2\Big|_0^1 z\Big|_0^1 = \frac{1}{2} \,. \tag{4.26}$$

4.3 Volume integrals

The definition of the volume integral is again taken as the limit of a sum of products as the size of the volume element tends to zero. One obvious difference though is that the element of volume is always a scalar. The possibilities are:

- $\int_V U(\mathbf{r})dV$ — scalar field; scalar integral.
- $\int_V \mathbf{a}dV$ — vector field; vector integral.

You have covered the scalar integral in the 1st year course, and the vector integral can be handled by taking the components of **a** and computing separate scalar integrals. That is, in Cartesian components,

$$\int_V \mathbf{a}dV = \hat{\mathbf{\imath}}\int_V a_1 dV + \hat{\mathbf{\jmath}}\int_V a_2 dV + \hat{\mathbf{k}}\int_V a_3 dV \; . \tag{4.27}$$

We shall return to think further about volume integrals after considering changing variables and curvilinear coordinates.

4.4 Changing variables: curvilinear coordinates

So far we have considered line, surface and volume integrals in Cartesian coordinates. But often the symmetry of the problem strongly hints that we should use another coordinate system. It is

- likely to be plane, cylindrical, or spherical polars,
- but can be something more exotic

Let us think about the problem quite generally first, before specializing to the polar family. The general name for any general "u, v, w" coordinate system is a **curvilinear coordinate system**.

Let us start with line integrals, as they raise all the issues but provide the simplest case.

4.4.1 What are the issues?

When you perform a line integral in Cartesian coordinates, we write

$$\mathbf{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}} \qquad \text{and} \;\Rightarrow \qquad d\mathbf{r} = dx\hat{\mathbf{\imath}} + dy\hat{\mathbf{\jmath}} + dz\hat{\mathbf{k}} \; . \tag{4.28}$$

It is convenient to use changes (dx, dy, dz) along the basis vectors $(\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}})$ because these are independent of each other. In Cartesians, we can be sure that length scales are properly handled because, as we saw in Lecture 3,

$$|d\mathbf{r}| = ds = \sqrt{dx^2 + dy^2 + dz^2} \; . \tag{4.29}$$

To perform the line integral, we are interested in obtaining an expression for $d\mathbf{r}$ as a sum of terms involving $du\hat{\mathbf{u}}$, $dv\hat{\mathbf{v}}$, and $dw\hat{\mathbf{w}}$, but the very first thing to stress is that

$$\left.\begin{aligned} \mathbf{r} &\neq u\hat{\mathbf{u}} + v\hat{\mathbf{v}} + w\hat{\mathbf{w}} \\ d\mathbf{r} &\neq du\hat{\mathbf{u}} + dv\hat{\mathbf{v}} + dw\hat{\mathbf{w}} \\ |d\mathbf{r}| = ds &\neq \sqrt{du^2 + dv^2 + dw^2} \end{aligned}\right\} \textbf{THESE ARE BAD} \tag{4.30}$$

The key thing is that the length scales have been lost, and must be restored.

4.4.2 Finding the length scales

As with almost all things in multivariate calculus, everything of importance appears with just two variables, and so let us think about a line integral in the plane, and transform from (x, y) to (u, v) coordinates.

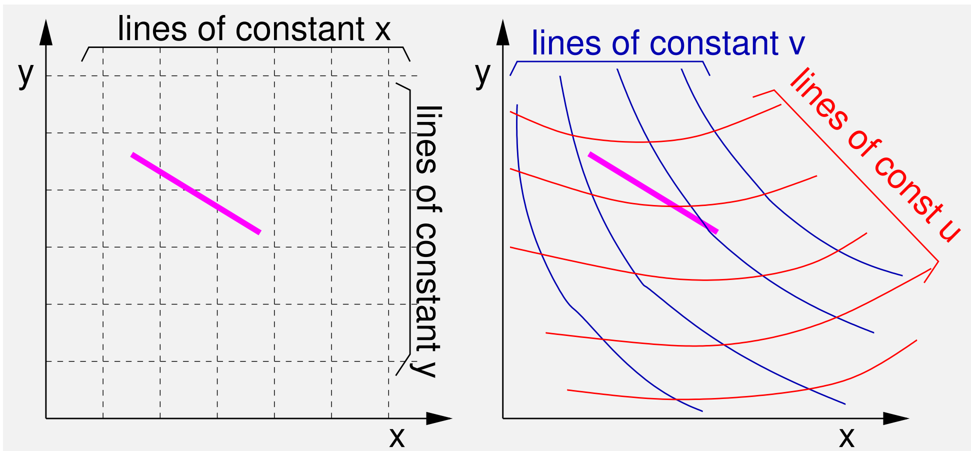


Figure 4.3: Lines of constant u and v appear as curves on the xy-plane. How do we express $d\mathbf{r}$?

We are told $x = x(u, v)$ and $y = y(u, v)$, so that

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \qquad \text{and} \qquad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\ . \tag{4.31}$$

Hence, as $\mathbf{r} = x(u, v)\hat{\mathbf{\imath}} + y(u, v)\hat{\mathbf{\jmath}}$, we can write

$$\begin{aligned} d\mathbf{r} &= \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)\hat{\mathbf{\imath}} + \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)\hat{\mathbf{\jmath}} & (4.32)\\ &= \left(\frac{\partial x}{\partial u}\hat{\mathbf{\imath}} + \frac{\partial y}{\partial u}\hat{\mathbf{\jmath}}\right)du \quad + \quad \left(\frac{\partial x}{\partial v}\hat{\mathbf{\imath}} + \frac{\partial y}{\partial v}\hat{\mathbf{\jmath}}\right)dv & (4.33)\\ &= \quad (h_u\ \hat{\mathbf{u}})\ du \qquad + \qquad (h_v\ \hat{\mathbf{v}})\ dv & (4.34)\end{aligned}$$

So, at a stroke, we have found expressions for $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, and have found the length scales h_u and h_v. These scales are called **metric coefficients**. They are the factors that turn the "d-whatevers" into proper lengths.

Because $\hat{\mathbf{u}}$ is a unit vector, if we square both sides of the expression $h_u\ \hat{\mathbf{u}} = (\partial x/\partial u\hat{\mathbf{\imath}} + \partial y/\partial u\hat{\mathbf{\jmath}})$ we find that

$$h_u = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2\right]^{1/2} \tag{4.35}$$

Because we can also write

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv \qquad (4.36)$$

we also have that

$$h_u \hat{\mathbf{u}} = \frac{\partial \mathbf{r}}{\partial u} \qquad \text{and} \qquad h_u = \left| \frac{\partial \mathbf{r}}{\partial u} \right| \qquad (4.37)$$

and similarly for v.

4.4.3 Now we can tie this in with our knowledge of tangents!

In Lecture 3 we discovered that $d\mathbf{r}/dp$ was a (non-unit) tangent to the curve $\mathbf{r}(p)$. Now suppose we wanted to write down the tangent to the v =constant curve. We know that $\mathbf{r} = x(u, v)\hat{\mathbf{\imath}} + y(u, v)\hat{\mathbf{\jmath}}$ and so

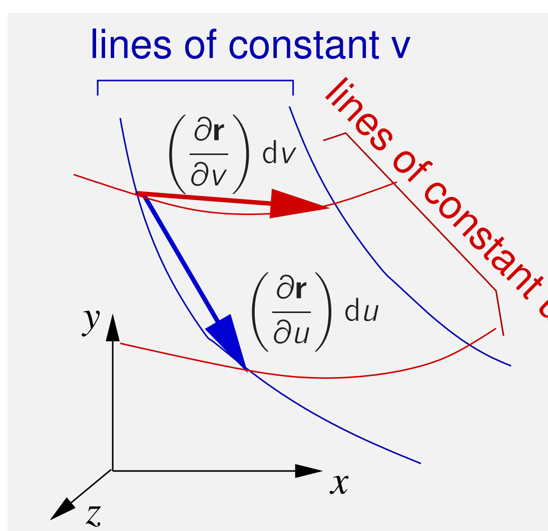
$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\hat{\mathbf{\imath}} + \frac{\partial y}{\partial u}\hat{\mathbf{\jmath}} \; . \qquad (4.38)$$

This is like $d\mathbf{r}/dp$ but is partial because there are two parameters and v is being held constant.

Clearly u is not arclength and $\partial\mathbf{r}/\partial u$ will not be a unit tangent, rather

$$\frac{\partial \mathbf{r}}{\partial u} = h_u \hat{\mathbf{u}} \qquad (4.39)$$

and similarly for $\hat{\mathbf{v}}$.
This is exactly what we derived before.



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These ideas extend to n-vectors without need for further proof, and so:

Summary: If

$$\mathbf{r} = x(u,v,w)\hat{\mathbf{i}} + y(u,v,w)\hat{\mathbf{j}} + z(u,v,w)\hat{\mathbf{k}} \tag{4.40}$$

then

$$d\mathbf{r} = h_u du\hat{\mathbf{u}} \ + \ h_v dv\hat{\mathbf{v}} \ + \ h_w dw\hat{\mathbf{w}} \tag{4.41}$$

where

$$h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right| \qquad h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right| \qquad h_w = \left|\frac{\partial \mathbf{r}}{\partial w}\right| \tag{4.42}$$

and, for example,

$$\left|\frac{\partial \mathbf{r}}{\partial u}\right| = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right]^{1/2} \tag{4.43}$$

4.4.4 Surface integrals and curvilinear coordinates

In the earlier surface integral example using a cube, to find the surface element with normal along $\hat{\mathbf{i}}$ we took a **vector product** of elements in the two orthogonal directions:

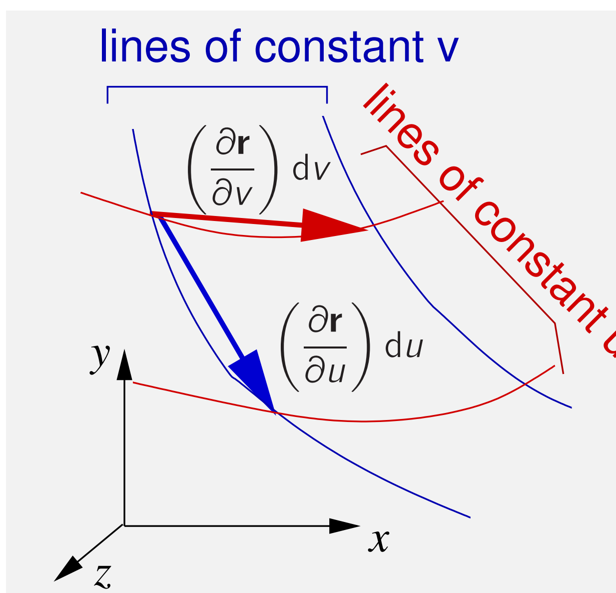
$$d\mathbf{S} = (dy\hat{\mathbf{j}}) \times (dz\hat{\mathbf{k}}) = dydz\hat{\mathbf{i}} \ . \tag{4.44}$$

The question now is can we use the same in curvilinear coordinates? To obtain a surface with normal along $\hat{\mathbf{w}}$, would we take vector products like $(du\hat{\mathbf{u}}) \times (dv\hat{\mathbf{v}})$? You can probably guess that this is nearly correct, but that length scales will trouble us ...

Looking at the elemental surface patch in u, v, w coordinates, we see that the surface element is planar (but not necessarily in the xy-plane).
So the surface element is

$$\begin{aligned} d\mathbf{S} &= \frac{\partial \mathbf{r}}{\partial u}du \times \frac{\partial \mathbf{r}}{\partial v}dv && (4.45)\\ &= h_u du\hat{\mathbf{u}} \times h_v dv\hat{\mathbf{v}} && (4.46)\end{aligned}$$

Note that the tile is a parallelogram, not a rectangle.



To summarize

The general 3D result for a surface patch is

$$d\mathbf{S} = h_u h_v du dv (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \tag{4.47}$$

For an **orthogonal** curvilinear coord system $\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{w}}$, and

$$d\mathbf{S} = h_u h_v du dv \hat{\mathbf{w}} \tag{4.48}$$

A note about Jacobians

Interestingly, if we deal with the change between variables (x, y) and (u, v) in the plane, we arrive at the familiar Jacobian.

$$\begin{aligned} d\mathbf{S} &= dxdy(\hat{\mathbf{\imath}} \times \hat{\mathbf{\jmath}}) = dxdy\hat{\mathbf{k}} && (4.49)\\ &= \left(\frac{\partial x}{\partial u}\hat{\mathbf{\imath}} + \frac{\partial y}{\partial u}\hat{\mathbf{\jmath}}\right) \times \left(\frac{\partial x}{\partial v}\hat{\mathbf{\imath}} + \frac{\partial y}{\partial v}\hat{\mathbf{\jmath}}\right) dudv && (4.50)\\ &= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right) dudv\ \hat{\mathbf{k}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv\ \hat{\mathbf{k}} && (4.51) \end{aligned}$$

(But note that for non-vector integration in two variables, vector signs are unimportant and, as you'll remember, you take the *modulus of the Jacobian* as the area scale factor.)

4.4.5 Curvilinear co-ordinates and volume integrals

If we transform to curvilinear coordinates u, v, w from x, y, z, what is the size of the volume element?

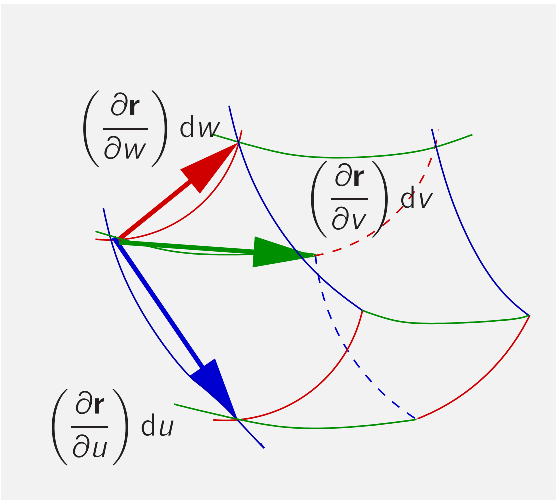


Figure 4.4:

It is the volume of a parallelopiped, which in an earlier lecture we saw was given by the *scalar triple product*. Hence

$$dV = \left(\frac{\partial \mathbf{r}}{\partial u}du \times \frac{\partial \mathbf{r}}{\partial v}dv\right) \cdot \frac{\partial \mathbf{r}}{\partial w}dw \quad = \quad h_u h_v h_w \;\; du\ dv\ dw \;\; (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}} \tag{4.52}$$

Recalling that

$$\frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial x}{\partial u}\hat{\mathbf{\imath}} + \frac{\partial y}{\partial u}\hat{\mathbf{\jmath}} + \frac{\partial z}{\partial u}\hat{\mathbf{k}}\right) \tag{4.53}$$

the scalar triple product is just the Jacobian:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \tag{4.54}$$

To summarize

General 3D results:

Either

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \;\; du\ dv\ dw \tag{4.55}$$

or

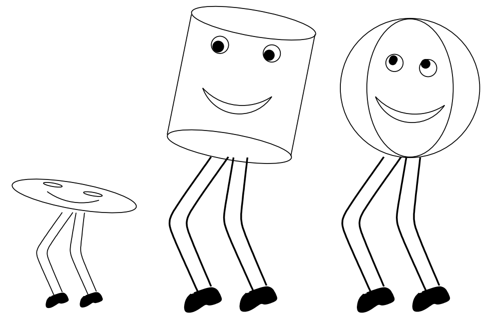
$$dV = h_u h_v h_w \;\; du\ dv\ dw \;\; (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}} \tag{4.56}$$

Short cut if you are sure the system is orthogonal

$$dV = h_u h_v h_w \;\; du\ dv\ dw \tag{4.57}$$

4.5 The Polars

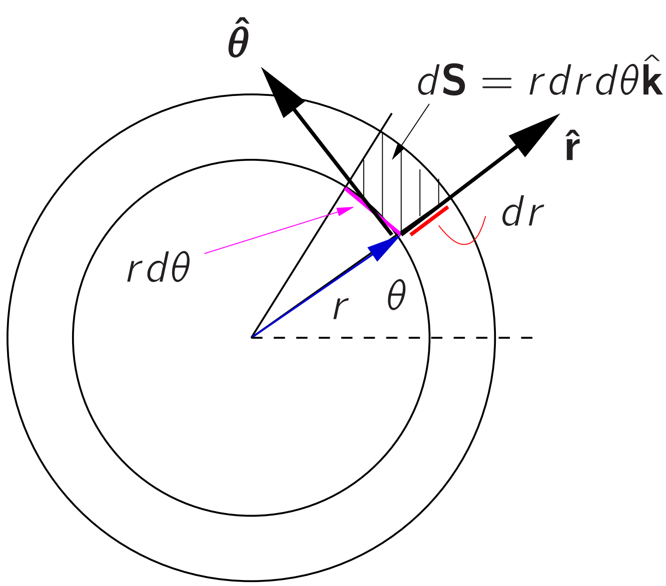
Some curvilinear coordinate systems are orthogonal, meaning that $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are mutually perpendicular. The family of polar coordinates are three such. Below, we apply the general theory to these in turn.



4.6 Plane polars: an orthogonal curvi coord system

Starting from the position vector, we can now work out the orthogonal vectors and metric coefficients

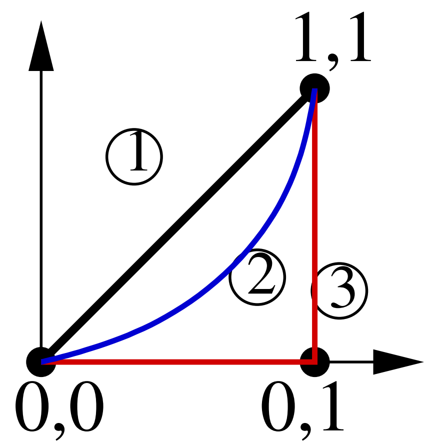
$$
\begin{aligned}
\mathbf{r} &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}} \qquad (4.58)\\
h_r\hat{\mathbf{r}} &= \frac{\partial\mathbf{r}}{\partial r} = (\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})\\
h_\theta\hat{\boldsymbol{\theta}} &= \frac{\partial\mathbf{r}}{\partial\theta} = (-r\sin\theta\hat{\mathbf{i}} + r\cos\theta\hat{\mathbf{j}})\\
\Rightarrow h_r &= \left|\frac{\partial\mathbf{r}}{\partial r}\right| = |\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}| = 1\\
h_\theta &= \left|\frac{\partial\mathbf{r}}{\partial\theta}\right| = |-r\sin\theta\hat{\mathbf{i}} + r\cos\theta\hat{\mathbf{j}}| = r\\
\hat{\mathbf{r}} &= (\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})\\
\hat{\boldsymbol{\theta}} &= (-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})\\
\Rightarrow d\mathbf{r} &= h_r dr\hat{\mathbf{r}} + h_\theta d\theta\hat{\boldsymbol{\theta}} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}}.\\
\text{and } d\mathbf{S} &= h_r h_\theta dr d\theta(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}) = rdrd\theta\hat{\mathbf{k}}\ .
\end{aligned}
$$



♣ Example

NB! This is not a very sensible example to turn into plane polars, but as we've done it Cartesians we know what the answer is.

Q:
Using plane polars, repeat path 3 of the earlier line integral example $\int \mathbf{F}\cdot d\mathbf{r}$ where the a force $\mathbf{F} = x^2y\hat{\mathbf{i}} + xy^2\hat{\mathbf{j}}$ acts on a body at it moves between $(0,0)$ and $(1,0)$ then from $(1,0)$ to $(1,1)$.



A: First change the functions and the vectors from Cartesian to plane polars:

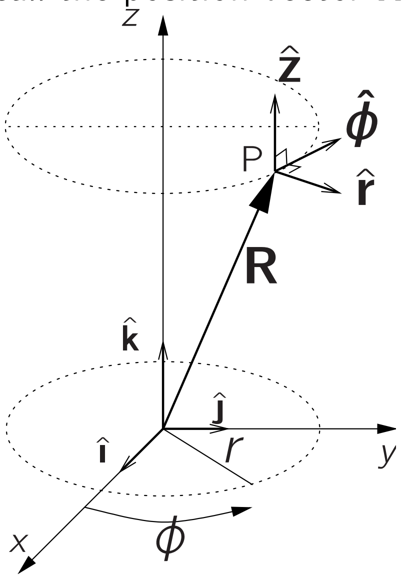
$$\begin{aligned} \mathbf{F} &= r^3\cos\theta\sin\theta(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}) = r^3\cos\theta\sin\theta\hat{\mathbf{r}} \\ d\mathbf{r} &= dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} \\ \Rightarrow \mathbf{F}\cdot d\mathbf{r} &= r^3\cos\theta\sin\theta dr \end{aligned} \tag{4.59}$$

Must break the path into two. Along the first part of the path, the integrand is zero as $\sin\theta = 0$. Along the second part, $r = 1/\cos\theta$, and hence

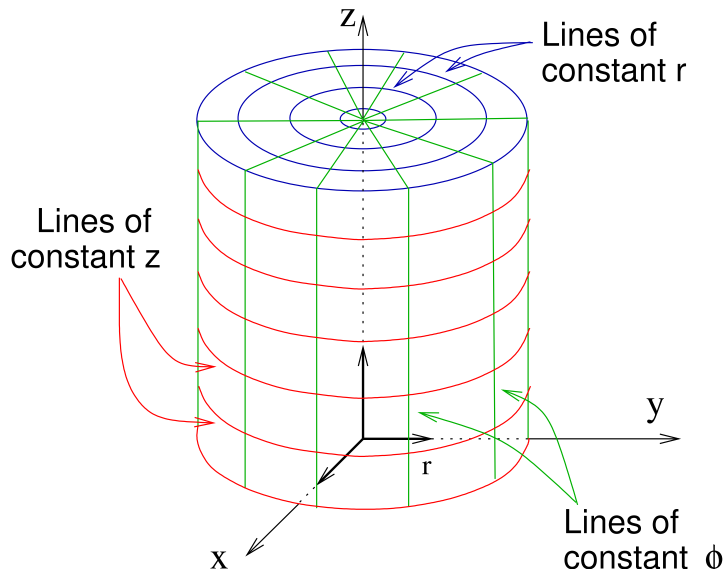
$$\begin{aligned} I &= \int_{r=0,\theta=0}^{r=1,\theta=0} r^3\cos\theta\sin\theta dr + \int_{r=1,\theta=0}^{r=\sqrt{2},\theta=\pi/4} r^3\cos\theta\sin\theta dr \\ &= 0 + \int_{\theta=0}^{\theta=\pi/4} \frac{1}{\cos^3\theta}\cos\theta\sin\theta\frac{\sin\theta}{\cos^2\theta}d\theta \\ &= \int_{\theta=0}^{\theta=\pi/4} \frac{\sin^2\theta}{\cos^4\theta}d\theta \qquad [\textbf{subst} \quad t = \tan\theta] = \int_0^1 t^2 dt = 1/3 . \end{aligned} \tag{4.60}$$

4.7 Cylindrical polar co-ordinates

This adds the cartesian co-ordinate z to plane polar co-ordinates in order to specify position in three dimensions. **NOTE:** The co-ordinate r is measured perpendicularly from the z axis, and this can cause confusion with the position vector. To avoid this we will call the position vector $\mathbf{R}$.



(a) Quantities



(b) Iso lines

We work through the equations in brief first,

$$
\begin{aligned}
x &= r\cos\phi\,, \quad y = r\sin\phi\,, \quad z = z &(4.61)\\
\Rightarrow \mathbf{R} &= r\cos\phi\hat{\mathbf{\imath}} + r\sin\phi\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}\\
h_r\hat{\mathbf{r}} &= \frac{\partial \mathbf{r}}{\partial r} = (\cos\phi\hat{\mathbf{\imath}} + \sin\phi\hat{\mathbf{\jmath}})\\
h_\phi\hat{\boldsymbol{\phi}} &= \frac{\partial \mathbf{r}}{\partial \phi} = (-r\sin\phi\hat{\mathbf{\imath}} + r\cos\phi\hat{\mathbf{\jmath}})\\
h_z\hat{\mathbf{z}} &= \frac{\partial \mathbf{r}}{\partial z} = \hat{\mathbf{k}}\\
\Rightarrow h_r &= 1 \text{ and } \hat{\mathbf{r}} = \cos\phi\hat{\mathbf{\imath}} + \sin\phi\hat{\mathbf{\jmath}}\\
h_\phi &= r \text{ and } \hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{\imath}} + \cos\phi\hat{\mathbf{\jmath}}\\
h_z &= 1 \text{ and } \hat{\mathbf{z}} = \hat{\mathbf{k}}\\
\Rightarrow d\mathbf{R} &= dr\;\hat{\mathbf{r}} + r d\phi\;\hat{\boldsymbol{\phi}} + dz\;\hat{\mathbf{z}}\\
\text{and } d\mathbf{S}_r &= h_\phi h_z d\phi dz(\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}}) = r\;d\phi\;dz\;\hat{\mathbf{r}}\\
d\mathbf{S}_\phi &= h_z h_r dz dr(\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = dz\;dr\;\hat{\boldsymbol{\phi}}\\
d\mathbf{S}_z &= h_r h_\phi dr d\phi(\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}}) = r dr d\phi\hat{\mathbf{z}}\\
dV &= r\;dr\;d\phi\;dz
\end{aligned}
$$

4.7.1 Detail: Surface integrals in cylindrical polars

Recall that in Cartesians for a surface element with normal along x or $\hat{\mathbf{i}}$ we $d\mathbf{S}_x = dydz\hat{\mathbf{j}} \times \hat{\mathbf{k}} = dydz\hat{\mathbf{i}}$.

We now know that we must insert scale parameters in curvilinear coordinates.

In cylindrical polars, the two most used surface area elements are given by:

For surfaces of constant r:

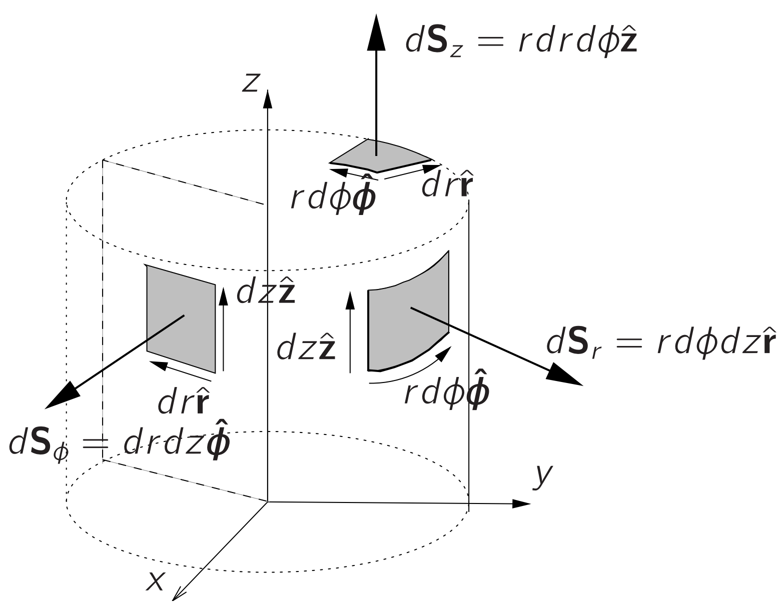
$$d\mathbf{S}_r = h_\phi h_z d\phi dz(\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}}) = r d\phi dz \hat{\mathbf{r}} \ . \tag{4.62}$$

For surfaces of constant z:

$$d\mathbf{S}_z = h_r h_\phi dr d\phi(\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}}) = r dr d\phi \hat{\mathbf{z}} \tag{4.63}$$

If your cylinder is cut open, you may also need the third element $d\mathbf{S}_\phi$ for surfaces of constant ϕ

$$d\mathbf{S}_\phi = h_z h_r dz dr(\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = dz dr \hat{\boldsymbol{\phi}} \ . \tag{4.64}$$



4.7.2 Detail: Volume integrals in cylindrical polars

In Cartesian coordinates a volume element is given by $dV = dxdydz$. Recall that the volume of a parallelopiped is given by the scalar triple product of the vectors which define it (see section **??**). Thus the formula above can be derived (even though it is "obvious") as: $dV = dx\hat{\mathbf{\imath}} \cdot (dy\hat{\mathbf{\jmath}} \times dz\hat{\mathbf{k}}) = dxdydz$ since the basis set is orthonormal.

In cylindrical polars a volume element is given by (see Fig. 4.5b):

$$dV = h_r dr\hat{\mathbf{r}} \cdot (h_\phi d\phi\hat{\boldsymbol{\phi}} \times h_z dz\hat{\mathbf{z}}) = r dr\ d\phi dz\ . \qquad (4.65)$$

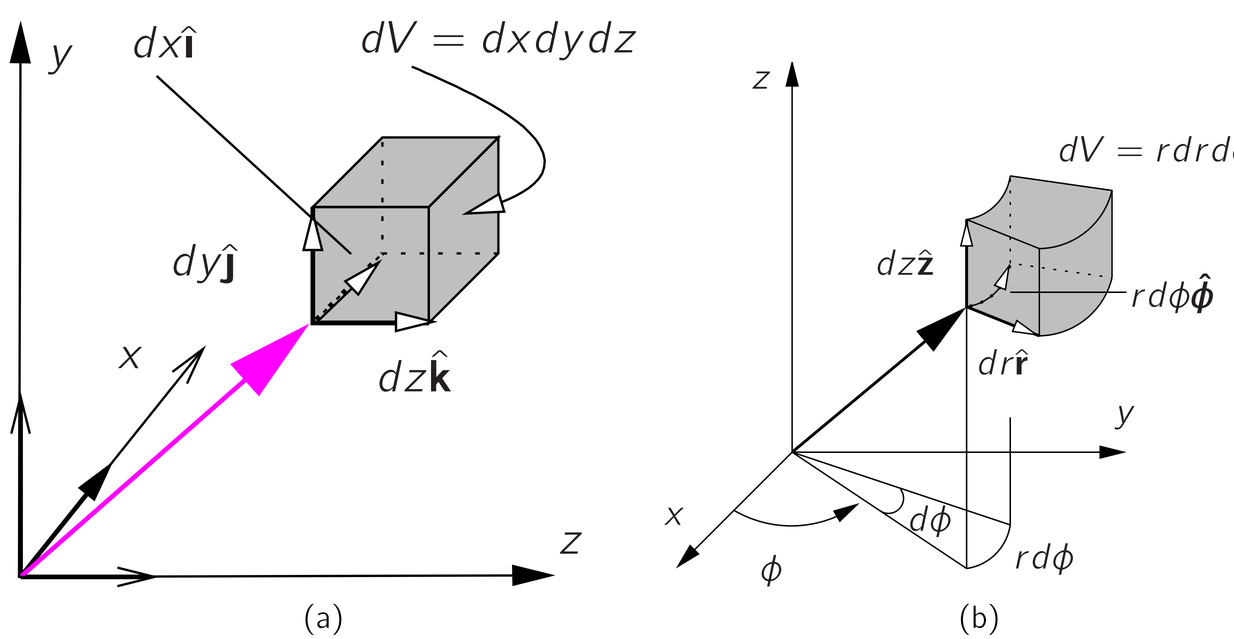


Figure 4.5: Volume elements dV in (a) Cartesian coordinates; (b) Cylindrical polar coordinates

Note also that this volume, because it is a scalar triple product, can be written as a determinant:

$$dV = \begin{vmatrix} \hat{\mathbf{r}}dr \\ \hat{\boldsymbol{\phi}}rd\phi \\ \hat{\mathbf{z}}dz \end{vmatrix} = \begin{vmatrix} (\partial\mathbf{R}/\partial r)dr \\ (\partial\mathbf{R}/\partial\phi)d\phi \\ (\partial\mathbf{R}/\partial z)dz \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} drd\phi dz$$

where the equality on the right-hand side follows from the definitions of

$$\hat{\mathbf{r}} = \partial\mathbf{R}/\partial r = \frac{\partial x}{\partial r}\hat{\mathbf{\imath}} + \frac{\partial y}{\partial r}\hat{\mathbf{\jmath}} + \frac{\partial z}{\partial r}\hat{\mathbf{k}} \qquad (4.66)$$

etc. **Remember** we are denoting the position vector by **R**.

♣ Example: line integral in cylindrical coordinates

Q

Evaluate $\oint_C \mathbf{a}\cdot d\mathbf{R}$, where $\mathbf{a} = x^3\hat{\mathbf{j}} - y^3\hat{\mathbf{i}} + x^2y\hat{\mathbf{k}}$ and C is the circle of radius r in the $z = 0$ plane, centred on the origin.

A

In this case our cylindrical coordinates effectively reduce to plane polars since the path of integration is a circle in the $z = 0$ plane, but let's persist with the full set of coordinates anyway; the $\hat{\mathbf{k}}$ component of $\mathbf{a}$ will play no role (it is normal to the path of integration and therefore disappears as seen below).

On the circle of interest

$$\mathbf{a} = r^3(-\sin^3\phi\hat{\mathbf{i}} + \cos^3\phi\hat{\mathbf{j}} + \cos^2\phi\sin\phi\hat{\mathbf{k}}) \tag{4.67}$$

In general, $d\mathbf{R} = h_r dr\hat{\mathbf{r}} + h_\phi d\phi\hat{\boldsymbol{\phi}} + h_z dz\hat{\mathbf{z}}$. But on the chosen path $dr = 0$ and $dz = 0$. Hence

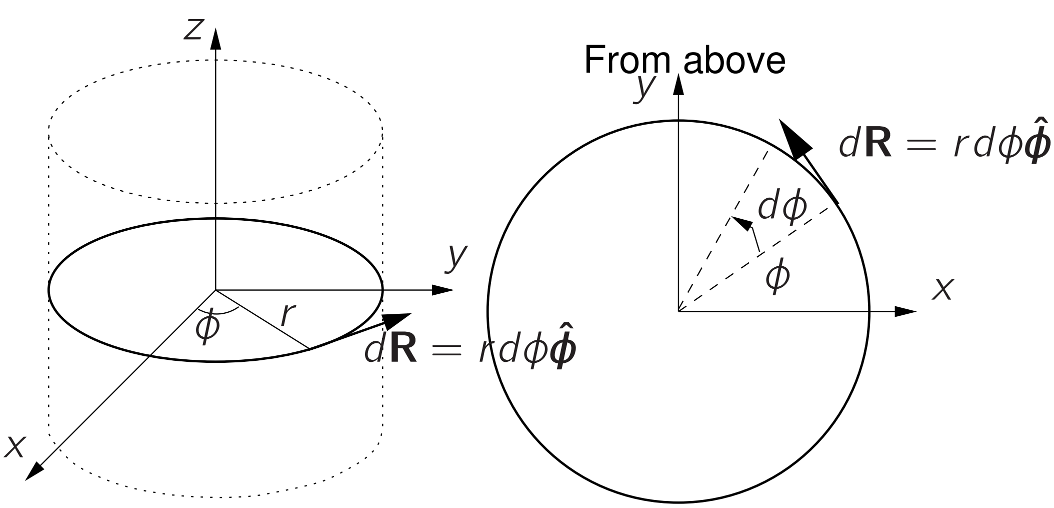
$$d\mathbf{R} = h_\phi d\phi\hat{\boldsymbol{\phi}} = r d\phi\hat{\boldsymbol{\phi}} = r d\phi(-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}) \tag{4.68}$$

so that

$$\oint_C \mathbf{a}\cdot d\mathbf{R} = \int_0^{2\pi} r^4(\sin^4\phi + \cos^4\phi)d\phi = \frac{3\pi}{2}r^4 \tag{4.69}$$

since

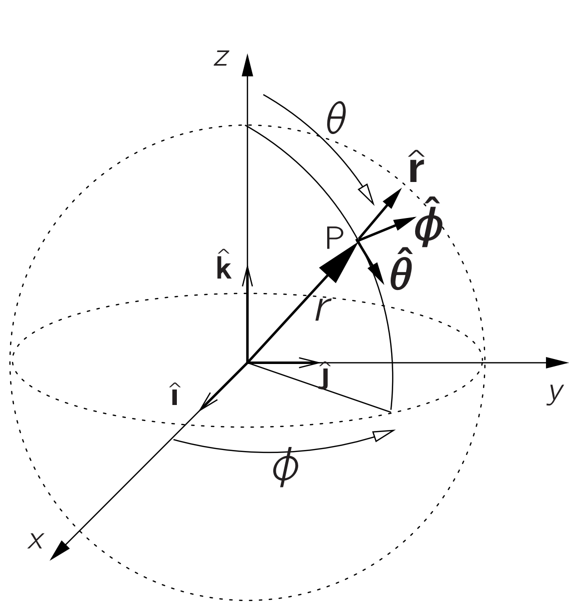
$$\int_0^{2\pi}\sin^4\phi d\phi = \int_0^{2\pi}\cos^4\phi d\phi = \frac{3\pi}{4} \tag{4.70}$$

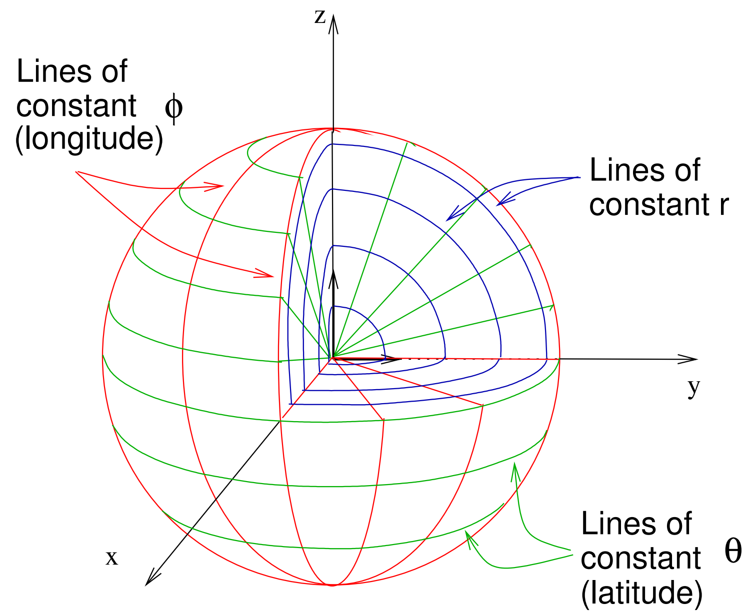


4.8 Spherical polar co-ordinates

Much of the development for spherical polars is similar to that for cylindrical polars. As shown below, a point in space P having cartesian coordinates x, y, z can be expressed in terms of spherical polar coordinates, r, θ, ϕ as follows:

$$\mathbf{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}} \quad = r\sin\theta\cos\phi\hat{\mathbf{\imath}} + r\sin\theta\sin\phi\hat{\mathbf{\jmath}} + r\cos\theta\hat{\mathbf{k}} \tag{4.71}$$





We work through the equations in brief first:

$$\begin{aligned}
x &= r\sin\theta\cos\phi\ , \quad y = r\sin\theta\sin\phi\ , \quad z = r\cos\theta \qquad (4.72)\\
\mathbf{r} &= r\sin\theta\cos\phi\hat{\mathbf{\imath}} + r\sin\theta\sin\phi\hat{\mathbf{\jmath}} + r\cos\theta\hat{\mathbf{k}}\\
\Rightarrow h_r\hat{\mathbf{r}} &= \partial\mathbf{r}/\partial r =\\
h_\theta\hat{\boldsymbol{\theta}} &= \partial\mathbf{r}/\partial\theta =\\
h_\phi\hat{\boldsymbol{\phi}} &= \partial\mathbf{r}/\partial\phi =\\
\Rightarrow h_r &= 1\ , \quad h_\theta = r, \quad h_\phi = r\sin\theta\\
\Rightarrow \hat{\mathbf{r}} &= \sin\theta\ \cos\phi\hat{\mathbf{\imath}} + \sin\theta\ \sin\phi\hat{\mathbf{\jmath}} + \cos\theta\hat{\mathbf{k}}\\
\hat{\boldsymbol{\theta}} &= \cos\theta\ \cos\phi\hat{\mathbf{\imath}} + \cos\theta\ \sin\phi\hat{\mathbf{\jmath}} - \sin\theta\hat{\mathbf{k}}\\
\hat{\boldsymbol{\phi}} &= -\sin\theta\hat{\mathbf{\imath}} + \cos\phi\ \hat{\mathbf{\jmath}}\\
\Rightarrow d\mathbf{r} &= dr\ \hat{\mathbf{r}} + rd\theta\ \hat{\boldsymbol{\theta}} + r\sin\theta\ d\phi\ \hat{\boldsymbol{\phi}}\\
d\mathbf{S}_r &= h_\theta h_\phi d\theta d\phi(\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}}) = r^2\ \sin\theta\ d\theta\ d\phi\ \hat{\mathbf{r}} \qquad \text{on spherical surface}\\
d\mathbf{S}_\theta &= h_\phi h_r d\phi dr(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{r}}) = r\sin\theta\ d\phi dr\hat{\boldsymbol{\theta}} \qquad \text{on conical surface}\\
d\mathbf{S}_\phi &= h_r h_\theta dr d\theta(\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}) = r\ drd\theta\hat{\boldsymbol{\phi}} \qquad \text{on planar hemisphere surface}\\
dV &= r^2\ \sin\theta\ dr\ d\theta\ d\phi
\end{aligned}$$

4.8.1 Detail: Surface integrals in spherical polars

The most useful surface element in spherical polars is that tangent to surfaces of constant r (see Fig. 4.6). This surface element $d\mathbf{S}_r$ is given by

$$d\mathbf{S}_r = h_\theta d\theta\hat{\boldsymbol{\theta}} \times h_\phi d\phi\hat{\boldsymbol{\phi}} = r^2 \sin\theta d\theta d\phi\hat{\mathbf{r}} \tag{4.73}$$

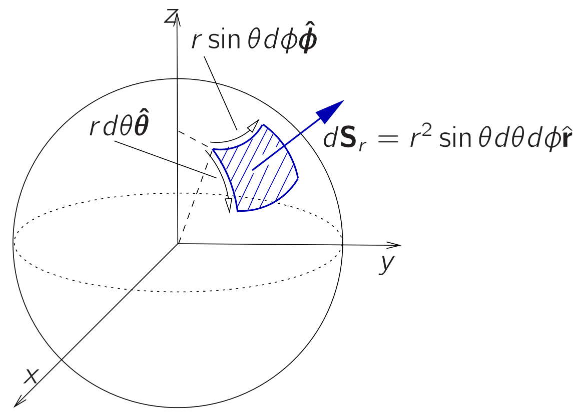


Figure 4.6: Surface element $d\mathbf{S}$ in spherical polar coordinates

♣ Example: surface integral in spherical polars

Q Evaluate $\int_S \mathbf{a}\cdot d\mathbf{S}$, where $\mathbf{a} = z^3\hat{\mathbf{k}}$ and S is the sphere of radius A centred on the origin.

A On the surface of the sphere:

$$\mathbf{a} = A^3 cos^3\theta\hat{\mathbf{k}}\ , \qquad d\mathbf{S} = A^2\sin\theta\ d\theta\ d\phi\hat{\mathbf{r}} \tag{4.74}$$

Hence

$$\int_S \mathbf{a}\cdot d\mathbf{S} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} A^3 cos^3\theta\ A^2\sin\theta\ [\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}]\ d\theta d\phi \tag{4.75}$$

$$\downarrow$$

$$= A^5\int_0^{2\pi} d\phi \int_0^{\pi} cos^3\theta\sin\theta[\cos\theta]\ d\theta$$

$$= 2\pi A^5\frac{1}{5}\left[-\cos^5\theta\right]_0^{\pi} = \frac{4\pi A^5}{5}$$

4.8.2 Detail: Volume integrals in spherical polars

In spherical polars a volume element is given by (see Fig. 4.7):

$$dV = h_r dr\hat{\mathbf{r}}\cdot(h_\theta d\theta\hat{\boldsymbol{\theta}} \times h_\phi d\phi\hat{\boldsymbol{\phi}}) = r^2\sin\theta dr d\theta d\phi\ . \tag{4.76}$$

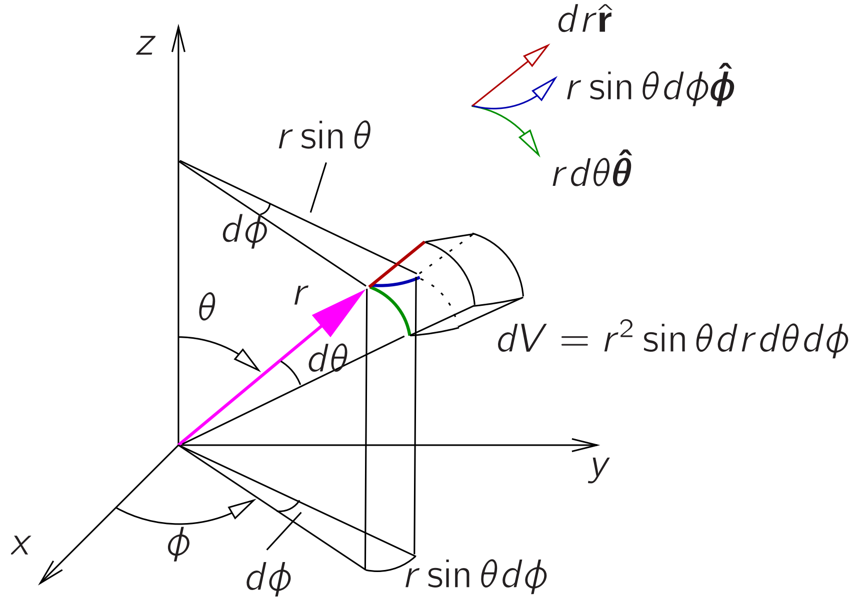


Figure 4.7: Volume element dV in spherical polar coordinates

It can also be written of course using the Jacobian, but this is left as an exercise for the reader.