
A1 Vector Algebra and Calculus

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Course Page

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Overview

This course is concerned chiefly with the properties of vectors which are related to physical processes in 3 spatial dimensions.

It starts by reviewing and, perhaps, developing your knowledge of vector algebra and geometry, but soon moves on to consider new material by applying calculus of single variables to individual vectors and to vector relationships. An important area here will be to understand how to describe curves in 3D spaces and how to perform integration along a curve.

The course then moves on to consider calculus of several variables applied to both scalar and vector fields. To give you a feeling for the issues, suppose you were interested in the temperature T of water in a river. Temperature T is a *scalar*, and will certainly be a function of a position vector $\mathbf{x} = (x, y, z)$ and may also be a function of time t : $T = T(\mathbf{x}, t)$. It is a scalar field. The heat flows generated by this temperature field will in general be non-uniform, and must be described by a vector field.

Now let's dive into a flow. At each point \mathbf{x} in the stream, at each time t , there will be a stream velocity $\mathbf{v}(\mathbf{x}, t)$. The local stream velocity can be viewed directly using modern techniques such as laser Doppler anemometry, or traditional techniques such as throwing twigs in. The point now is that \mathbf{v} is a function that has the same four input variables as temperature did, but its output result is a vector. We may be interested in places \mathbf{x} where the stream suddenly accelerates, or vortices where the stream curls around dangerously. That is, we will be interested in finding the acceleration of the stream, the gradient of its velocity. We may be interested in the magnitude of the acceleration (a scalar). Equally, we may be interested in the acceleration as a vector, so that we can apply Newton's law and figure out the force. This is the stuff of vector calculus.

Syllabus

Vector algebra: scalar and vector products; scalar and vector triple products; geometric applications. Differentiation of a vector function; scalar and vector fields. Gradient, divergence and curl - definitions and physical interpretations; product formulae; curvilinear coordinates. Gauss' and Stokes' theorems and evaluation of integrals over lines, surfaces and volumes. Derivation of continuity equations and Laplace's equation in Cartesian, cylindrical and spherical coordinate systems.

Course Content

- Revision: scalar and vector products; product, vector product.
- Triple products, multiple products, applications to geometry.
- Differentiation and integration of vector functions of a single variable. Space curves
- Curvilinear coordinate systems. Line, surface and volume integrals.
- Vector operators.
- Vector Identities.
- Gauss' and Stokes' Theorems.
- Engineering Applications.

Learning Outcomes

You should be comfortable with expressing systems (especially those in 2 and 3 dimensions) using vector quantities and manipulating these vectors without necessarily going back to some underlying coordinates.

You should have a sound grasp of the concept of a vector field, and be able to link this idea to descriptions various physical phenomena.

You should have a good intuition of the physical meaning of the various vector calculus operators and the important related theorems. You should be able to interpret the formulae describing physical systems in terms of this intuition.

References

Although these notes cover the material you need to know you should, wider reading is essential. Different explanations and different diagrams in books will give you the perspective to glue everything together, and further worked examples give you the confidence to tackle the tute sheets.

- J Heading, "Mathematical Methods in Science and Engineering", 2nd ed., Ch.13, (Arnold).
- G Stephenson, "Mathematical Methods for Science Students", 2nd ed., Ch.19, (Longman).
- E Kreyszig, "Advanced Engineering Mathematics", 6th ed., Ch.6, (Wiley).
- K F Riley, M. P. Hobson and S. J. Bence, "Mathematical Methods for the Physics and Engineering" Chs.6, 8 and 9, (CUP).
- A J M Spencer, et. al. "Engineering Mathematics", Vol.1, Ch.6, (Van Nostrand Reinhold).
- H M Schey, "Div, Grad, Curl and all that", Norton

Course WWW Pages

Pdf copies of these notes, pdf copies of the lecture slides, the tutorial sheets, FAQs etc will be accessible from

www.robots.ox.ac.uk/~dwm/Courses/2VA

Just the notes and the tute sheets get put on weblearn.

Lecture 1

Vector Algebra

Notation

In these notes we will use a bold font $\mathbf{a}, \mathbf{b}, \dots, \boldsymbol{\alpha}, \boldsymbol{\beta}, \dots$ to represent vectors and the corresponding non-bold font $a, b, \dots, \alpha, \beta, \dots$ as their magnitudes. Unit vectors will wear hats $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \dots, \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \dots$

The unit vector along Cartesian x, y, z axes are $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. Later we will require unit vectors in cylindrical and spherical coordinates. These will be $\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ and $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$. (Note that HLT uses $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$ for these unit vectors.) On the odd occasion we use $\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}}$ as an orthogonal trio. Please distinguish $\hat{\mathbf{l}}$ from $\hat{\mathbf{i}}$.

In your written work, underline the vector symbol $\underline{a}, \underline{\omega}$ and be **meticulous** about doing so. Chaos ensues if you can't distinguish vectors from scalars in expressions like $r^2\mathbf{r}$.

1.1 Vectors

Many physical quantities, such a mass, time, temperature are scalars, and are fully specified by one number or magnitude. But others, such velocity, electric field, and force, are specified by both magnitude and direction, requiring two numbers in a 2D planar world and three in 3D.

These are vectors — but there are three slightly different types:

- **Free vectors:** In many situations *only* the magnitude and direction of a vector are important, and we can *translate* them at will (with 3 degrees of freedom for a vector in 3-dimensions).
- **Sliding vectors:** In mechanics the line of action of a force is often important for deriving moments. The force vector can slide with 1 degree of freedom.
- **Bound or position vectors:** When describing lines, curves etc in space, it is obviously important that the origin and head of the vector are not translated

about arbitrarily. The origins of position vectors all coincide at an overall origin O .

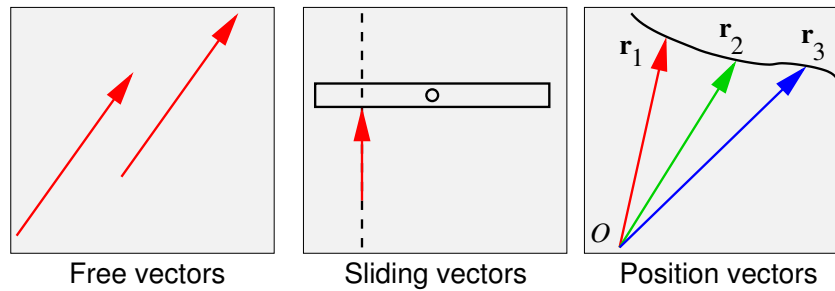


Figure 1.1:

The chief power of vector representation (and one that is often not appreciated) is that analysis is freed from the restriction of arbitrarily imposed coordinate frames. As a simple example, if two free vectors are equal we need only say that their magnitudes and directions are equal, and that can be done with a drawing that is independent of any coordinate system. (For a more sophisticated example, see the last question on 2A1B.) Try to spot things in the notes that are independent of coordinate system.

However, coordinate systems are ultimately useful when performing calculation, so it is useful to introduce the idea of vector components.

1.1.1 Components in a coordinate frame

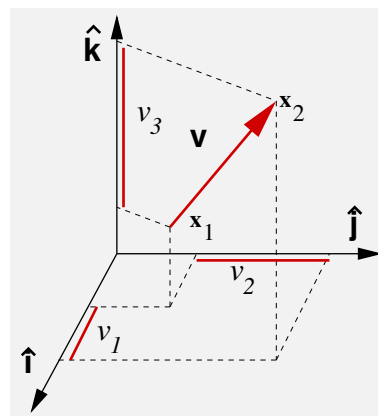


Figure 1.2: Vector components.

A vector can be represented by the values of its components in a number of different directions, the number equal to the dimension of the vector space. It is most convenient if those directions are mutually orthogonal (but, as we see later, they do not have to be). Referring to Fig. 1.2, we would write $\mathbf{v} = [v_1, v_2, v_3] = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$. As $\hat{\mathbf{i}} = [1, 0, 0]$, $\hat{\mathbf{j}} = [0, 1, 0]$, and $\hat{\mathbf{k}} = [0, 0, 1]$, we can also write $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$.

1.1.2 Vector equality

Two free vectors are said to be equal iff their lengths and directions are the same. If we use a coordinate frame, we might say that corresponding components of the two vectors must be equal. This definition of equality will also do for position vectors, but for sliding vectors we must add that the line of action must be identical too.

1.1.3 Vector magnitude and unit vectors

Provided we use an orthogonal coordinate system, the magnitude of a 3-vector is $v = |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. To find the unit vector in the direction of \mathbf{v} , simply divide by its magnitude $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$.

1.1.4 Vector Addition and subtraction

Addition follows the parallelogram construction of Figure 1.3(a). Subtraction ($\mathbf{a} - \mathbf{b}$) is defined as the addition ($\mathbf{a} + (-\mathbf{b})$). It is useful to remember that the vector $\mathbf{a} - \mathbf{b}$ goes from \mathbf{b} to \mathbf{a} . The following results follow immediately from the above definition of vector addition:

- (a) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity) (Figure 1.3(a))
- (b) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = \mathbf{a} + \mathbf{b} + \mathbf{c}$ (associativity) (Figure 1.3(b))
- (c) $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$, where the zero vector is $\mathbf{0} = [0, 0, 0]$.
- (d) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

It is obvious that in terms of components vector addition involves the addition of the corresponding components. That is $\mathbf{a} + \mathbf{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$.

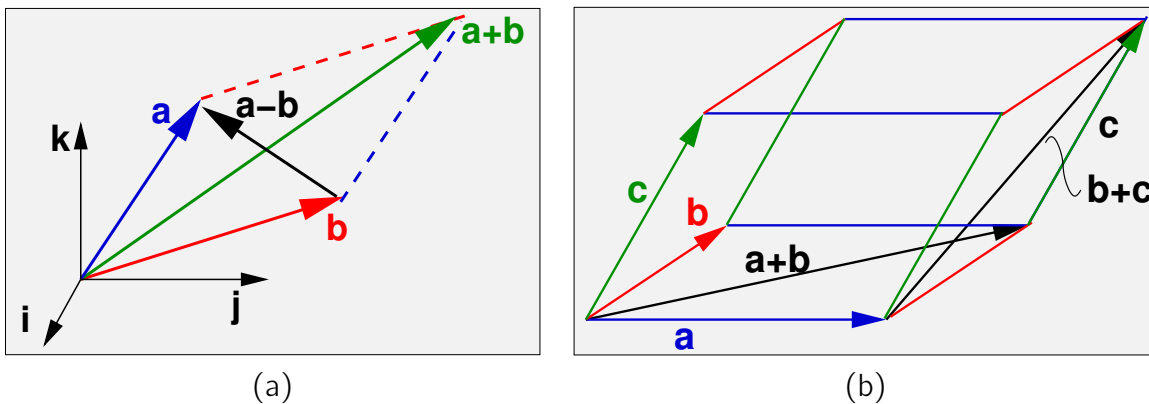


Figure 1.3: (a) Addition of two vectors is commutative. Note that the coordinate frame is irrelevant. (b) subtraction of vectors; (c) Addition of three vectors is associative.

1.1.5 Multiplication of a vector by a scalar. (NOT the scalar product!)

Just as for matrices, multiplication of a vector \mathbf{a} by a scalar c is defined as multiplication of each component by c , so that $c\mathbf{a} = [ca_1, ca_2, ca_3]$.

It follows that $|c\mathbf{a}| = \sqrt{(ca_1)^2 + (ca_2)^2 + (ca_3)^2} = |c||\mathbf{a}|$.

The direction of the vector will reverse if c is negative, but otherwise is unaffected. (By the way, a vector where the sign is uncertain is called a director.)

♣ Example

Q: Coulomb's law states that the electrostatic force on charged particle Q due to another charged particle q_1 is

$$\mathbf{F} = K \frac{Qq_1}{r^2} \hat{\mathbf{r}} \quad (1.1)$$

where \mathbf{r} is the vector from q_1 to Q and $\hat{\mathbf{r}}$ is the unit vector in that same direction. (Note that the rule "unlike charges attract, like charges repel" is built into this formula.) The force between two particles is not modified by the presence of other charged particles.

Hence write down an expression for the force on Q at \mathbf{R} due to N charges q_i at \mathbf{r}_i .

A: The vector from q_i to Q is $\mathbf{R} - \mathbf{r}_i$. The unit vector in that direction is $(\mathbf{R} - \mathbf{r}_i)/|\mathbf{R} - \mathbf{r}_i|$, so the resultant force is

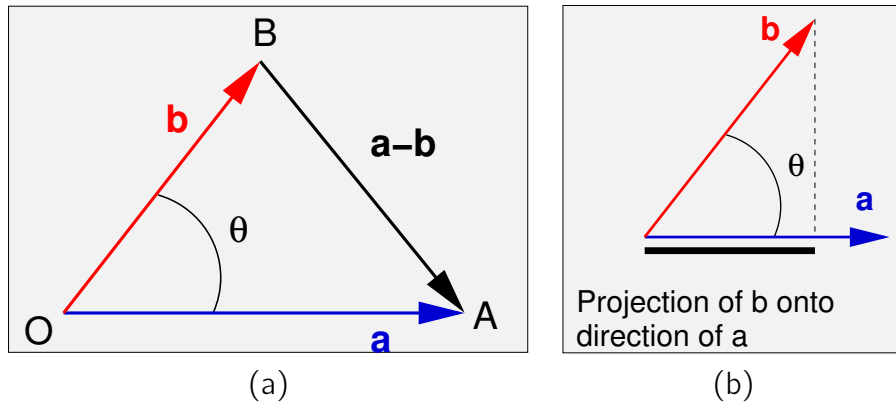
$$\mathbf{F}(\mathbf{R}) = \sum_{i=1}^N K \frac{Qq_i}{|\mathbf{R} - \mathbf{r}_i|^3} (\mathbf{R} - \mathbf{r}_i) \quad (1.2)$$

Note that $\mathbf{F}(\mathbf{R})$ is a vector *field*.

1.2 Scalar, dot, or inner product

This is a product of two vectors results in a scalar quantity and is defined as follows for 3-component vectors: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$. Note that $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2 = a^2$. The following laws of multiplication follow immediately from the definition:

- (a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutativity)
- (b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (distributivity with respect to vector addition)
- (c) $(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda\mathbf{b})$ scalar multiple of a scalar product of two vectors

Figure 1.4: (a) Cosine rule. (b) Projection of \mathbf{b} onto \mathbf{a} .

1.2.1 Geometrical interpretation of scalar product

Consider the square magnitude of the vector $\mathbf{a} - \mathbf{b}$. By the rules of the scalar product, this is

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2(\mathbf{a} \cdot \mathbf{b}) \\ &= a^2 + b^2 - 2(\mathbf{a} \cdot \mathbf{b}) \end{aligned} \quad (1.3)$$

But, by the cosine rule for the triangle OAB (Figure 1.4a), the length AB^2 is given by

$$|\mathbf{a} - \mathbf{b}|^2 = a^2 + b^2 - 2ab \cos \theta \quad (1.4)$$

where θ is the angle between the two vectors. It follows that

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta, \quad (1.5)$$

which is independent of the co-ordinate system used, and that $|\mathbf{a} \cdot \mathbf{b}| \leq ab$. Conversely, the cosine of the angle between vectors \mathbf{a} and \mathbf{b} is given by $\cos \theta = \mathbf{a} \cdot \mathbf{b} / ab$.

1.2.2 Projection of one vector onto another

Another way of describing the scalar product is as the product of the magnitude of one vector and the component of the other in the direction of the first, since $b \cos \theta$ is the component of \mathbf{b} in the direction of \mathbf{a} and vice versa (Figure 1.4b).

Projection is particularly useful when the second vector is a unit vector — $\mathbf{a} \cdot \hat{\mathbf{i}}$ is the component of \mathbf{a} in the direction of $\hat{\mathbf{i}}$.

Notice that if we wanted the **vector component** of \mathbf{b} in the direction of \mathbf{a} we would write

$$(\mathbf{b} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}} = \frac{(\mathbf{b} \cdot \mathbf{a})\mathbf{a}}{a^2}. \quad (1.6)$$

In the particular case $\mathbf{a} \cdot \mathbf{b} = 0$, the angle between the two vectors is a right angle and the vectors are said to be mutually orthogonal or perpendicular — neither vector has any component in the direction of the other.

An orthonormal coordinate system is characterised by $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$; and $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$.

1.2.3 A scalar product is an “inner product”

In linear algebra it usual to assume a vector \mathbf{v} is a column vector, so that multiplying out $\mathbf{M}\mathbf{v}_1 = \mathbf{v}_2$ follows the usual conventions. For much of vector algebra there is no need to worry at all — we will tend to write vectors as row vectors $\mathbf{a} = [a_1, a_2, a_3]$ simply because its take up less room on the page.

One of the few places where you might worry is with the equivalence of the scalar product and the inner inner product more commonly used in linear algebra. Defined as $\mathbf{a}^T \mathbf{b}$ when vectors are column vectors as

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = [a_1, a_2, a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 . \quad (1.7)$$

It is not something over which to lose sleep!

♣ Examples

Q1:

A force \mathbf{F} is applied to an object as it moves by a small displacement $\delta \mathbf{r}$. What work is done on the object by the force?

A1:

The work done is equal to the component of force in the direction of the displacement multiplied by the displacement itself. This is just a scalar product:

$$\delta W = \mathbf{F} \cdot \delta \mathbf{r} . \quad (1.8)$$

Q2:

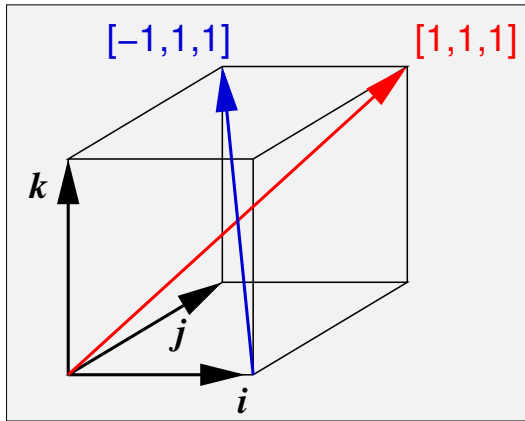
A cube has four diagonals, connecting opposite vertices. What is the angle between an adjacent pair?

A2:

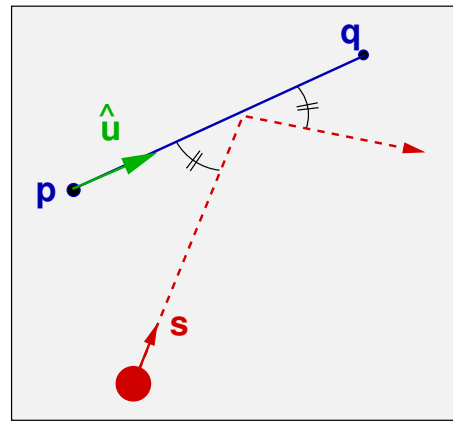
You could plod through using Pythagoras' theorem to find the length of the diagonal from cube vertex to cube centre, and perhaps you should to check the following answer.

The directions of the diagonals are $[\pm 1, \pm 1, \pm 1]$. The ones shown in the figure are $[1, 1, 1]$ and $[-1, 1, 1]$. The angle is thus

$$\theta = \cos^{-1} \frac{[1, 1, 1] \cdot [-1, 1, 1]}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{-1^2 + 1^2 + 1^2}} = \cos^{-1} \frac{1}{3} \quad (1.9)$$



Cube example



Pinball example

Q3:

A pinball moving in a plane with velocity \mathbf{s} bounces (in a purely elastic impact) from a baffle whose endpoints are \mathbf{p} and \mathbf{q} . What is the velocity vector after the bounce?

A3: Rather than imposing a new coordinate frame, it is best to refer to a natural frame that involves $\hat{\mathbf{u}}$ as a principal direction.

$$\hat{\mathbf{u}} = (\mathbf{q} - \mathbf{p}) / |\mathbf{q} - \mathbf{p}|. \quad (1.10)$$

The vector component of \mathbf{s} along $\hat{\mathbf{u}}$ is $\mathbf{s}_{\parallel} = (\mathbf{s} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$, so the vector component perpendicular is just \mathbf{s} minus the \mathbf{s}_{\parallel}

$$\mathbf{s}_{\perp} = \mathbf{s} - (\mathbf{s} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}. \quad (1.11)$$

Physics tells us that after the impact, the component of velocity in the direction of the baffle is unchanged and the component normal to the baffle is reversed:

$$\mathbf{s}_{\text{after}} = (\mathbf{s}_{\parallel} - \mathbf{s}_{\perp}) = (\mathbf{s} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} - (\mathbf{s} - (\mathbf{s} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}) = 2(\mathbf{s} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} - \mathbf{s}. \quad (1.12)$$

1.3 Vector or cross product

The vector product of two vectors \mathbf{a} and \mathbf{b} is denoted by $\mathbf{a} \times \mathbf{b}$ and is defined as follows

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\hat{\mathbf{i}} + (a_3b_1 - a_1b_3)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}. \quad (1.13)$$

It is MUCH more easily remembered in terms of the pseudo-determinant

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (1.14)$$

where the top row consists of the vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ rather than scalars.

Since a determinant with two equal rows has value zero, it follows that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$. It is also easily verified that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$, so that $\mathbf{a} \times \mathbf{b}$ is orthogonal (perpendicular) to both \mathbf{a} and \mathbf{b} , as shown in Figure 1.5.

The magnitude of the vector product can be obtained by showing that $|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = a^2b^2$ from which it follows that $|\mathbf{a} \times \mathbf{b}| = ab \sin \theta$, which is again independent of the co-ordinate system used. This is left as an exercise.

Unlike the scalar product, the vector product does not satisfy commutativity but is in fact anti-commutative, in that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. Moreover the vector product does not satisfy the associative law of multiplication either since, as we shall see later $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

Since the vector product is known to be orthogonal to both the vectors which form the product, it merely remains to specify its sense with respect to these vectors. Assuming that the co-ordinate vectors form a right-handed set in the order $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ it can be seen that the sense of the the vector product is also right handed, i.e the vector product has the same sense as the co-ordinate system used.

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{\mathbf{k}}. \quad (1.15)$$

In practice, figure out the direction from a right-handed screw twisted from the first to second vector as shown in Figure 1.5(a).

1.3.1 Geometrical interpretation of vector product

The magnitude of the vector product $(\mathbf{a} \times \mathbf{b})$ is equal to the area of the parallelogram whose sides are parallel to, and have lengths equal to the magnitudes of, the vectors \mathbf{a} and \mathbf{b} (Figure 1.5b). Its direction is perpendicular to the parallelogram.

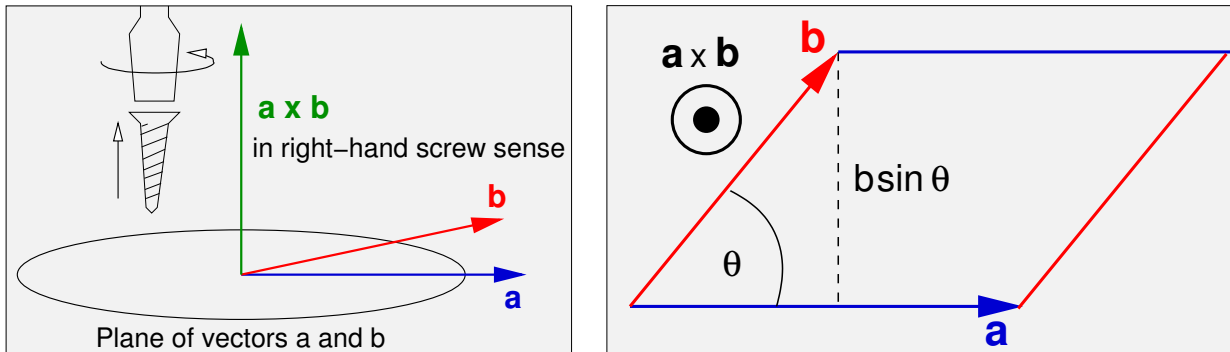


Figure 1.5: (a) The vector product is orthogonal to both \mathbf{a} and \mathbf{b} . Twist from first to second and move in the direction of a right-handed screw. (b) Area of parallelogram is $ab \sin \theta$.

♣ Example

Q: \mathbf{g} is vector from A [1,2,3] to B [3,4,5].

$\hat{\mathbf{a}}$ is the unit vector in the direction from O to A.

Find $\hat{\mathbf{b}}$, a UNIT vector along $\mathbf{g} \times \hat{\mathbf{a}}$

Verify that $\hat{\mathbf{b}}$ is perpendicular to $\hat{\mathbf{a}}$.

Find $\hat{\mathbf{c}}$, the third member of a right-handed coordinate set $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$.

A:

$$\mathbf{g} = [3, 4, 5] - [1, 2, 3] = [2, 2, 2] \quad (1.16)$$

$$\hat{\mathbf{a}} = \frac{1}{\sqrt{14}}[1, 2, 3] \quad (1.17)$$

$$\mathbf{g} \times \hat{\mathbf{a}} = \frac{1}{\sqrt{14}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \frac{1}{\sqrt{14}}[2, -4, 2] \quad (1.18)$$

Hence

$$\hat{\mathbf{b}} = \frac{1}{\sqrt{24}}[2, -4, 2] = \frac{1}{\sqrt{6}}[1, -2, 1] \quad (1.19)$$

and

$$\hat{\mathbf{c}} = \hat{\mathbf{a}} \times \hat{\mathbf{b}} \quad (1.20)$$

1.4 Vector components with a non-orthogonal basis set

Earlier we revised the idea of describing a vector as a linear sum of orthogonal components: that is, $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$.

We also discussed components of a vector, and you will recognize v_1 as the component of \mathbf{v} in the direction of $\hat{\mathbf{i}}$ — that is, $v_1 = \mathbf{v} \cdot \hat{\mathbf{i}}$, and so on.

Now suppose we replaced $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ with a **non-orthogonal, non-unit and non-coplanar** set of vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. By combining suitable amounts of these we can reach any point in 3-space, so any vector must be expressible as

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} . \quad (1.21)$$

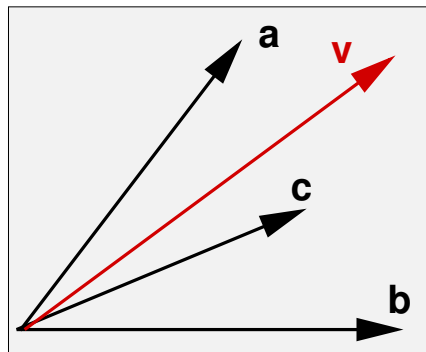


Figure 1.6: A basis set in 3D made up of three non-orthogonal, non-unit, non-coplanar vectors.

The question now is what is α ? You will be tempted to answer “the component of \mathbf{v} in the direction of \mathbf{a} ”. That is,

$$\alpha = (\mathbf{v} \cdot \hat{\mathbf{a}}) \quad \text{THIS IS WRONG} \quad (1.22)$$

The way to find α is to dot the RHS of the equation with a vector that is perpendicular to both \mathbf{b} and \mathbf{c} . The obvious choice is $(\mathbf{b} \times \mathbf{c})$, hence

$$\begin{aligned} \mathbf{v} \cdot (\mathbf{b} \times \mathbf{c}) &= \alpha\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \beta\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) + \gamma\mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}) \\ &= \alpha\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \end{aligned} \quad (1.23)$$

This is a scalar equation, so that

$$\alpha = \frac{\mathbf{v} \cdot (\mathbf{b} \times \mathbf{c})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})} \quad \text{and} \quad \beta = \frac{\mathbf{v} \cdot (\mathbf{c} \times \mathbf{a})}{\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})} \quad \gamma = \frac{\mathbf{v} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})} \quad (1.24)$$

Notice that the results are unique. Even though the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are not orthogonal one cannot substitute, say, a bit more \mathbf{b} and \mathbf{c} for a loss of a bit of \mathbf{a} .

Also notice that this general result specializes properly when the set is orthonormal. For example

$$\alpha = \frac{\mathbf{v} \cdot (\hat{\mathbf{j}} \times \hat{\mathbf{k}})}{\hat{\mathbf{i}} \cdot (\hat{\mathbf{j}} \times \hat{\mathbf{k}})} = \frac{\mathbf{v} \cdot \hat{\mathbf{i}}}{\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}} = \mathbf{v} \cdot \hat{\mathbf{i}} . \quad (1.25)$$

The expressions for α, β, γ in Eq. 1.24 fail when the denominator is zero. As we shall see, this occurs when the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar.

Lecture 2

Multiple Products. Geometry using Vectors

2.1 Scalar triple product

Using mixtures of the pairwise scalar product and vector product, it is possible to derive “triple products” between three vectors, and indeed n -products between n vectors.

This is the scalar product of a vector product and a third vector, i.e. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Using the pseudo-determinant expression for the vector product, we see that the scalar triple product can be represented as the true determinant

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (2.1)$$

You will recall that if you swap a pair of rows of a determinant, its sign changes; hence if you swap two pairs, its sign stays the same.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \begin{matrix} \text{1st SWAP} \\ + \end{matrix} \quad \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \quad \begin{matrix} \text{2nd SWAP} \\ - \end{matrix} \quad \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \begin{matrix} + \\ (2.2) \end{matrix}$$

This says that

- (i) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ (Called cyclic permutation.)
- (ii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$ and so on. (Called anti-cyclic permutation.)
- (iii) The fact that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ allows the scalar triple product to be written as $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$. This notation is not very helpful, and we will try to avoid it below.

2.1.1 Geometrical interpretation of scalar triple product

The scalar triple product gives the volume of the parallelepiped whose sides are represented by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} .

We saw earlier that the vector product ($\mathbf{a} \times \mathbf{b}$) has magnitude equal to the area of the base, and direction perpendicular to the base. The *component* of \mathbf{c} in this direction is equal to the height of the parallelepiped shown in Figure 2.1(a).

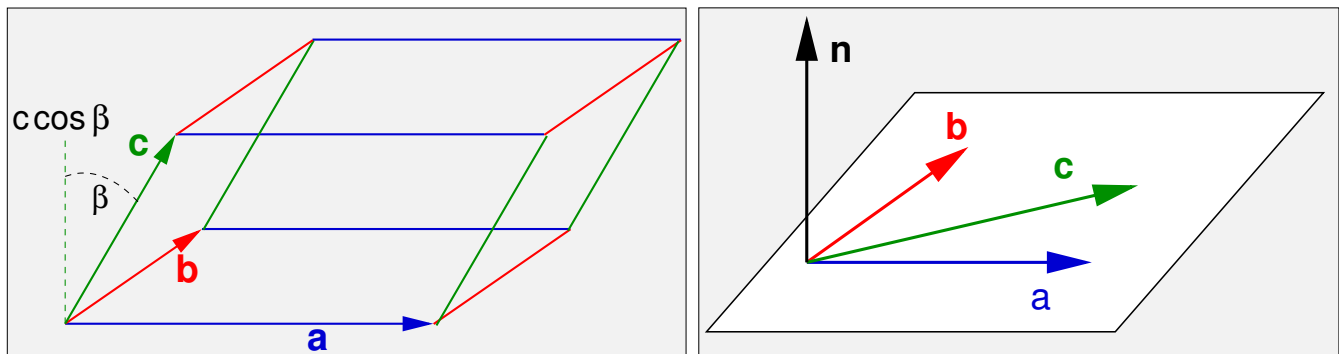


Figure 2.1: (a) Scalar triple product equals volume of parallelepiped. (b) Coplanarity yields zero scalar triple product.

2.1.2 Linearly dependent vectors

If the scalar triple product of three vectors is zero

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0 \quad (2.3)$$

then the vectors are **linearly dependent**. That is, one can be expressed as a linear combination of the others. For example,

$$\mathbf{a} = \lambda \mathbf{b} + \mu \mathbf{c} \quad (2.4)$$

where λ and μ are scalar coefficients.

You can see this immediately in two ways:

- The determinant would have one row that was a linear combination of the others. You'll remember that by doing row operations, you could reach a row of zeros, and so the determinant is zero.
- The parallelepiped would have zero volume if squashed flat. In this case all three vectors lie in a plane, and so any one is a linear combination of the other two. (Figure 2.1b.)

2.2 Vector triple product

This is defined as the vector product of a vector with a vector product, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Now, the vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ must be perpendicular to $(\mathbf{b} \times \mathbf{c})$, which in turn is perpendicular to both \mathbf{b} and \mathbf{c} . Thus $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ can have no component perpendicular to \mathbf{b} and \mathbf{c} , and hence must be coplanar with them. It follows that the vector triple product must be expressible as a linear combination of \mathbf{b} and \mathbf{c} :

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \lambda \mathbf{b} + \mu \mathbf{c} . \quad (2.5)$$

The values of the coefficients can be obtained by multiplying out in component form. Only the first component need be evaluated, the others then being obtained by symmetry. That is

$$\begin{aligned} (\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_1 &= a_2(\mathbf{b} \times \mathbf{c})_3 - a_3(\mathbf{b} \times \mathbf{c})_2 & (2.6) \\ &= a_2(b_1c_2 - b_2c_1) + a_3(b_1c_3 - b_3c_1) \\ &= (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1 \\ &= (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1 \\ &= (\mathbf{a} \cdot \mathbf{c})b_1 - (\mathbf{a} \cdot \mathbf{b})c_1 \end{aligned}$$

The equivalents must be true for the 2nd and 3rd components, so we arrive at the identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} . \quad (2.7)$$

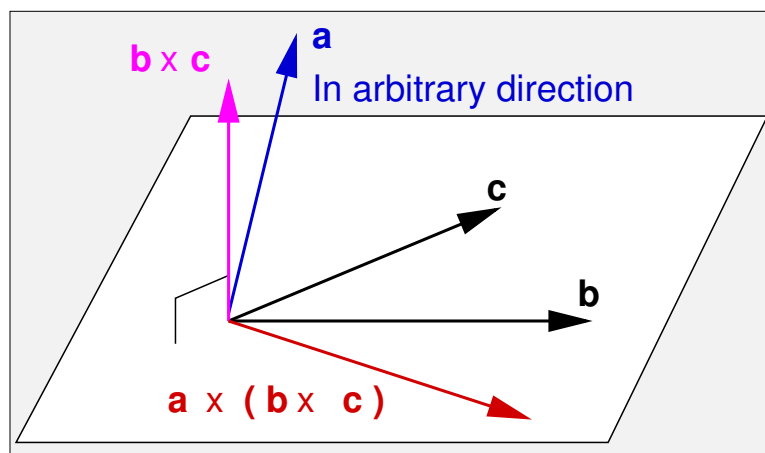


Figure 2.2: Vector triple product.

2.2.1 Projection using vector triple product

An example of the application of this formula is as follows. Suppose \mathbf{v} is a vector and we want its projection into the xy -plane. The component of \mathbf{v} in the z direction is $\mathbf{v} \cdot \hat{\mathbf{k}}$, so the projection we seek is $\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}$. Writing $\hat{\mathbf{k}} \leftarrow \mathbf{a}$, $\mathbf{v} \leftarrow \mathbf{b}$, $\hat{\mathbf{k}} \leftarrow \mathbf{c}$,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \quad (2.8)$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \hat{\mathbf{k}} \times (\mathbf{v} \times \hat{\mathbf{k}}) & = & (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}})\mathbf{v} - (\hat{\mathbf{k}} \cdot \mathbf{v})\hat{\mathbf{k}} \\ & = & \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} \end{array} \quad (2.9)$$

So, now we recognize that we have two equivalent expressions for the vector component of a vector \mathbf{v} perpendicular to the unit direction $\hat{\mathbf{k}}$:

$$\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} = \hat{\mathbf{k}} \times (\mathbf{v} \times \hat{\mathbf{k}}) . \quad (2.10)$$

(Hot stuff! But the expression $\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}$ is much easier to understand, and cheaper to compute!)

2.3 Other repeated products

Many combinations of vector and scalar products are possible, but we consider only one more, namely the vector quadruple product $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$. By regarding $\mathbf{a} \times \mathbf{b}$ as a single vector, we see that this vector must be representable as a linear combination of \mathbf{c} and \mathbf{d} . On the other hand, regarding $\mathbf{c} \times \mathbf{d}$ as a single vector, we see that it must also be a linear combination of \mathbf{a} and \mathbf{b} . This provides a means of expressing one of the vectors, say \mathbf{d} , as linear combination of the other three, as follows:

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\mathbf{c} - [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\mathbf{d} \\ &= [(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a}]\mathbf{b} - [(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b}]\mathbf{a} \end{aligned} \quad (2.11)$$

Hence

$$[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\mathbf{d} = [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\mathbf{a} + [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}]\mathbf{b} + [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\mathbf{c} \quad (2.12)$$

or

$$\mathbf{d} = \frac{[(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\mathbf{a} + [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}]\mathbf{b} + [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\mathbf{c}}{[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}] = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} . \quad (2.13)$$

This is much more complicated way of obtaining exactly the same result that we derived earlier!

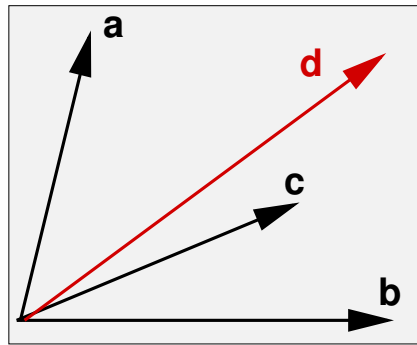


Figure 2.3: The projection of a (3-)vector onto a set of (3) basis vectors is unique. I.e. in $\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$, the set $\{\alpha, \beta, \gamma\}$ is unique.

2.4 Geometry using vectors: lines, planes

2.4.1 The equation of a line

The equation of the line passing through the point whose position vector is \mathbf{a} and lying in the direction of vector \mathbf{b} is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} \quad (2.14)$$

where λ is a scalar parameter. If you make \mathbf{b} a unit vector, $\mathbf{r} = \mathbf{a} + \lambda\hat{\mathbf{b}}$ then λ will represent metric length.

For a line defined by two points \mathbf{a}_1 and \mathbf{a}_2

$$\mathbf{r} = \mathbf{a}_1 + \lambda(\mathbf{a}_2 - \mathbf{a}_1) \quad (2.15)$$

or for the unit version

$$\mathbf{r} = \mathbf{a}_1 + \lambda \frac{(\mathbf{a}_2 - \mathbf{a}_1)}{|\mathbf{a}_2 - \mathbf{a}_1|} \quad (2.16)$$

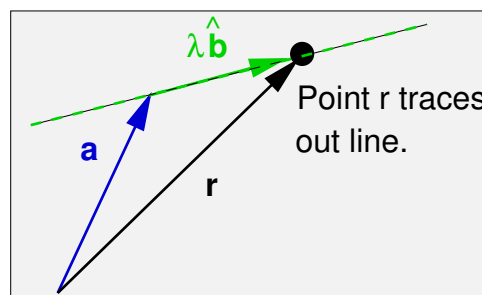


Figure 2.4: Equation of a line. With $\hat{\mathbf{b}}$ a unit vector, λ is in the length units established by the definition of \mathbf{a} .

2.4.2 The shortest distance from a point to a line

When working out the shortest distance from a point to a line it is usually sufficient to assume that this distance is found along the perpendicular to the line. First here, let's prove that too.

Referring to Figure 2.5(a) the vector \mathbf{p} from \mathbf{c} to any point on the line is $\mathbf{p} = \mathbf{a} + \lambda \hat{\mathbf{b}} - \mathbf{c} = (\mathbf{a} - \mathbf{c}) + \lambda \hat{\mathbf{b}}$ which has length squared $p^2 = (\mathbf{a} - \mathbf{c})^2 + \lambda^2 + 2\lambda(\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}}$. Rather than minimizing length, it is easier to minimize length-squared. The minimum is found when $d p^2 / d\lambda = 0$, ie when

$$\lambda = -(\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}}. \quad (2.17)$$

So the minimum length vector is

$$\mathbf{p} = (\mathbf{a} - \mathbf{c}) - ((\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}. \quad (2.18)$$

You might spot that is the component of $(\mathbf{a} - \mathbf{c})$ perpendicular to $\hat{\mathbf{b}}$ (as expected!). Furthermore, using the result of Section 2.2.1,

$$\mathbf{p} = \hat{\mathbf{b}} \times [(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}]. \quad (2.19)$$

Because $\hat{\mathbf{b}}$ is a unit vector, and is orthogonal to $[(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}]$, the modulus of the vector can be written rather more simply as just

$$p_{\min} = |(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}|. \quad (2.20)$$

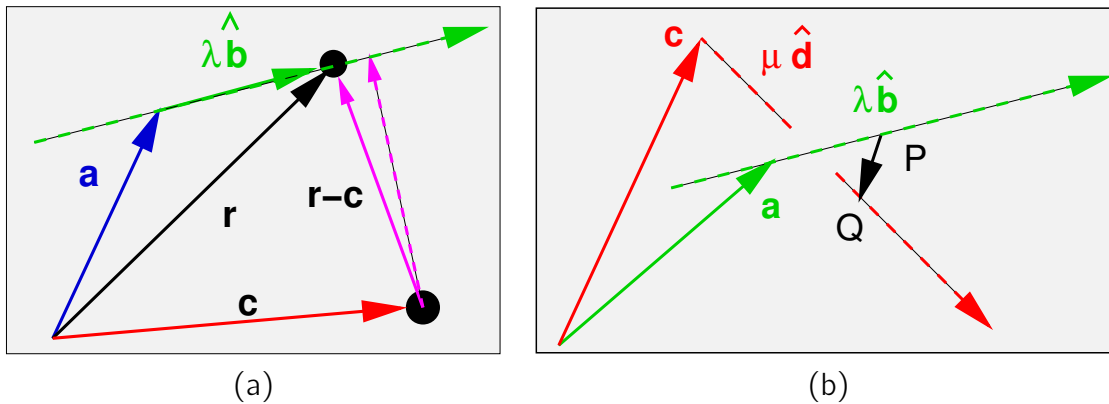


Figure 2.5: (a) Shortest distance point to line. (b) Shortest distance, line to line.

2.4.3 The shortest distance between two straight lines

If the shortest distance between a point and a line is along the perpendicular, then the shortest distance between the two straight lines $\mathbf{r} = \mathbf{a} + \lambda \hat{\mathbf{b}}$ and $\mathbf{r} = \mathbf{c} + \mu \hat{\mathbf{d}}$ must be found as the length of the vector which is mutually perpendicular to the lines.

The unit vector along the mutual perpendicular is

$$\hat{\mathbf{p}} = (\hat{\mathbf{b}} \times \hat{\mathbf{d}}) / |\hat{\mathbf{b}} \times \hat{\mathbf{d}}|. \quad (2.21)$$

(Yes, don't forget that $\hat{\mathbf{b}} \times \hat{\mathbf{d}}$ is NOT a unit vector. $\hat{\mathbf{b}}$ and $\hat{\mathbf{d}}$ are not orthogonal, so there is a $\sin \theta$ lurking!)

The minimum length is therefore the component of $\mathbf{a} - \mathbf{c}$ in this direction

$$p_{\min} = |(\mathbf{a} - \mathbf{c}) \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{d}}) / |\hat{\mathbf{b}} \times \hat{\mathbf{d}}||. \quad (2.22)$$

♣ Example

Q Two long straight pipes are specified using Cartesian co-ordinates as follows:

Pipe A has diameter 0.8 and its axis passes through points (2, 5, 3) and (7, 10, 8).

Pipe B has diameter 1.0 and its axis passes through the points (0, 6, 3) and (-12, 0, 9).

Determine whether the pipes need to be realigned to avoid intersection.

A Each pipe axis is defined using two points. The vector equation of the axis of pipe A is

$$\mathbf{r} = [2, 5, 3] + \lambda'[5, 5, 5] = [2, 5, 3] + \lambda[1, 1, 1]/\sqrt{3} \quad (2.23)$$

The equation of the axis of pipe B is

$$\mathbf{r} = [0, 6, 3] + \mu'[12, 6, 6] = [0, 6, 3] + \mu[-2, -1, 1]/\sqrt{6} \quad (2.24)$$

The perpendicular to the two axes has direction

$$[1, 1, 1] \times [-2, -1, 1] = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = [2, -3, 1] = \mathbf{p} \quad (2.25)$$

The length of the mutual perpendicular is

$$(\mathbf{a} - \mathbf{c}) \cdot \frac{[2, -3, 1]}{\sqrt{14}} = [2, -1, 0] \cdot \frac{[2, -3, 1]}{\sqrt{14}} = 1.87. \quad (2.26)$$

But the sum of the radii of the two pipes is $0.4 + 0.5 = 0.9$. Hence the pipes do not intersect.

2.4.4 The equation of a plane

There are a number of ways of specifying the equation of a plane.

1. If \mathbf{b} and \mathbf{c} are two non-parallel vectors (ie $\mathbf{b} \times \mathbf{c} \neq \mathbf{0}$), then the equation of the plane passing through the point \mathbf{a} and parallel to the vectors \mathbf{b} and \mathbf{c} may be written in the form

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c} \quad (2.27)$$

where λ, μ are scalar parameters. Note that \mathbf{b} and \mathbf{c} are free vectors, so don't have to lie in the plane (Figure 2.6(a).)

2. Figure 2.6(b) shows the plane defined by three non-collinear points \mathbf{a} , \mathbf{b} and \mathbf{c} in the plane (note that the vectors \mathbf{b} and \mathbf{c} are position vectors, not free vectors as in the previous case). The equation might be written as

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) \quad (2.28)$$

3. Figure 2.6(c) illustrates another description in terms of the unit normal to the plane $\hat{\mathbf{n}}$ and a point \mathbf{a} in the plane

$$\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}} . \quad (2.29)$$

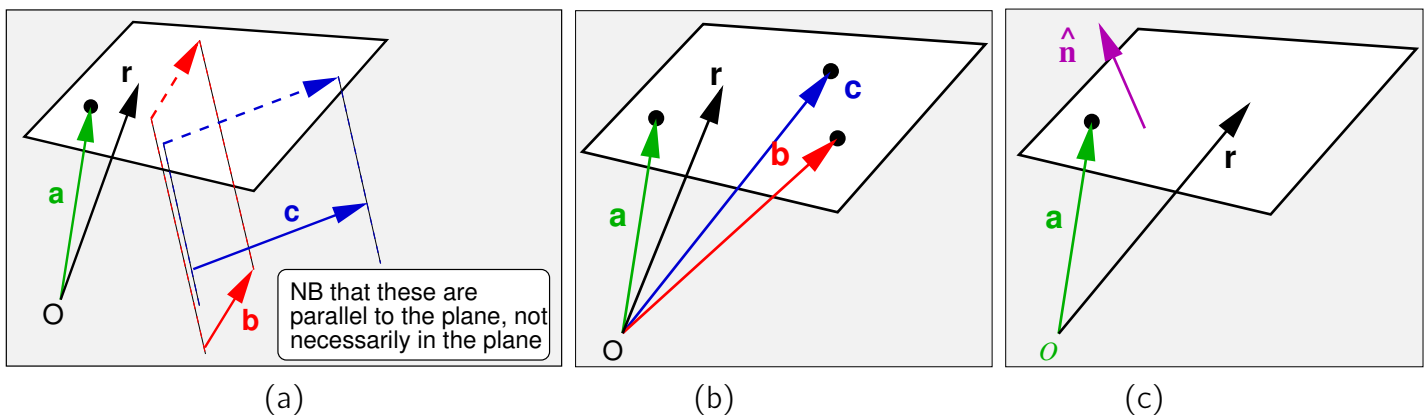


Figure 2.6: (a) Plane defined using point and two lines. (b) Plane defined using three points. (c) Plane defined using point and normal. Vector \mathbf{r} is the position vector of a general point in the plane.

2.4.5 The shortest distance from a point to a plane

The shortest distance from a point \mathbf{d} to the plane is along the perpendicular. Depending on how the plane is defined, this can be written as

$$D = |(\mathbf{d} - \mathbf{a}) \cdot \hat{\mathbf{n}}| \quad \text{or} \quad D = \frac{|(\mathbf{d} - \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{b} \times \mathbf{c}|} . \quad (2.30)$$

2.5 Rotation of vectors, in brief

In 3D, a rotation matrix R is a 3×3 matrix. The orthonormality condition $R^T = R^{-1}$ imposes 6 constraints on the matrix so that R has only 3 degrees of freedom. Its determinant is unity, its trace is $2 \cos \theta + 1$ where θ is the rotation angle and its eigenvalues are $(1, e^{\pm i\theta})$, where the eigenvalue 1 corresponds to the eigenvector which is the rotation axis $R\mathbf{a} = \mathbf{a}$.

The three degrees of freedom in one rotation matrix can be considered to lie in the direction of the rotation axis (2 d.o.f) and the rotation angle about it (1 d.o.f.). However, often it is easier to compose the overall rotation as a sequence of 3 “elementary” rotations about either fixed or successive principal axes (eg., x, y, z). There are 24 possibilities for choosing these combinations, subsets of which are called variously as Euler, Cardan or Tait-Bryan angles. One well-known example from aircraft dynamics is Yaw-Pitch-Roll where successive rotations are measured successive principal directions of an aircraft’s body.

The other issue to consider is whether the rotation is leaving the “world” in place and rotating the coordinates, or rotating the world within the same fixed coordinates.

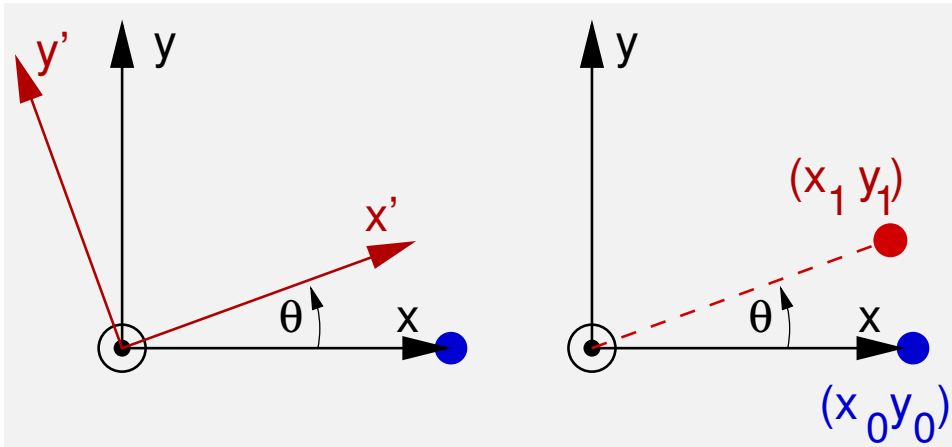


Figure 2.7: Left: The world is stationary, but the coordinate system is rotated. Right: The world is rotated within fixed coordinates.

For a rotation of θ in the rh sense about the z -axis, the first and second are

$$\mathbf{x}' = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{x}_1 = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_0 \quad (2.31)$$

Which sin term has the minus sign is easily determine using the trial point — $[1, 0, 0]$ in our case. For the rotating axes case, the fixed point must have coordinates $[x', y', z'] = [\cos \theta, -\sin \theta, 0]$ placing the minus sign as shown.

2.6 Angular velocity/acceleration and moments, in brief

A rotation can be represented by a vector whose direction is along the axis of rotation in the sense of a r-h screw, and whose magnitude is proportional to the size of the rotation (Fig. 2.8). The same idea can be extended to the derivatives, that is, angular velocity $\boldsymbol{\omega}$ and angular acceleration $\dot{\boldsymbol{\omega}}$.

Angular accelerations arise because of a moment (or torque) on a body. In mechanics, the moment of a force \mathbf{F} about a point Q is defined to have magnitude $M = Fd$, where d is the perpendicular distance between Q and the line of action L of \mathbf{F} .

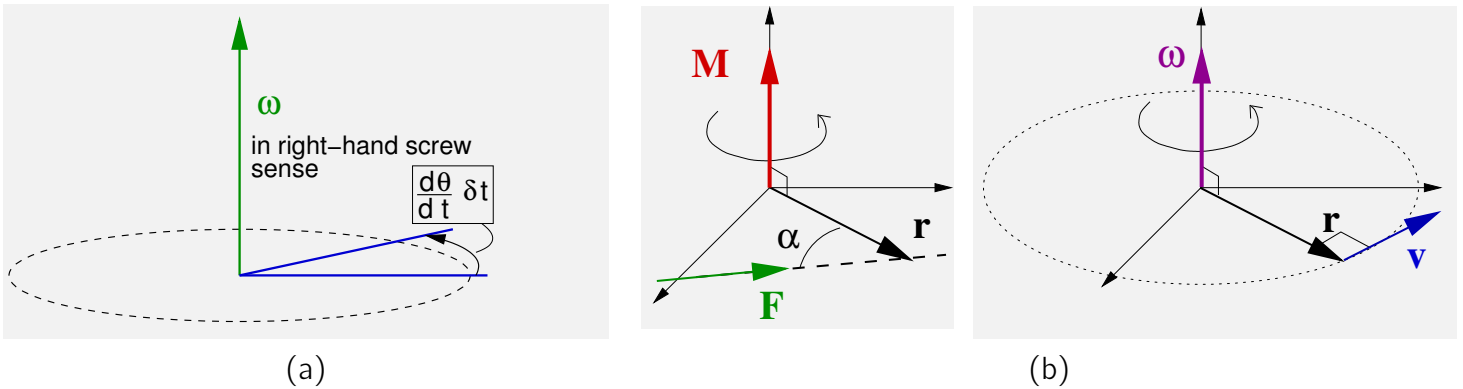


Figure 2.8: (a) The angular velocity vector $\boldsymbol{\omega}$ is along the axis of rotation and has magnitude equal to the rate of rotation. (b)

The vector equation for moment is

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \quad (2.32)$$

where \mathbf{r} is the vector from Q to *any* point on the line of action L of force \mathbf{F} . The resulting angular acceleration vector is in the same direction as the moment vector.

The instantaneous velocity of any point P on a rigid body undergoing pure rotation can be defined by a vector product as follows. The angular velocity vector $\boldsymbol{\omega}$ has magnitude equal to the angular speed of rotation of the body and with direction the same as that of the r-h screw. If \mathbf{r} is the vector OP , where the origin O can be taken to be any point on the axis of rotation, then the velocity \mathbf{v} of P due to the rotation is given, in both magnitude and direction, by the vector product $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

2.7 Solutions to general vector equations

Not all equations for a variable vector \mathbf{x} have a unique point solution, and often the locus is not a simple recognizable curve or region.

Examples of such equations are

$$\mathbf{x} = \mathbf{x} \times \mathbf{a} + \mathbf{b} \quad (2.33)$$

$$\mathbf{x} \cdot \mathbf{a} = \gamma. \quad (2.34)$$

A standard recipe for proceeding in such cases is as follows:

1. Decide upon two non-parallel vectors appearing in the vector relationship. These might be \mathbf{a} , \mathbf{b} and their vector product ($\mathbf{a} \times \mathbf{b}$).

If there is only one vector \mathbf{a} , say, then \mathbf{b} can be freely chosen, with the proviso that it is non-parallel to \mathbf{a} .

2. Generate $\mathbf{a} \times \mathbf{b}$.
3. Express the unknown vector \mathbf{x} as a linear combination of these vectors

$$\mathbf{x} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{a} \times \mathbf{b}$$

where λ, μ, ν are scalars to be found.

4. Substitute the above expression for \mathbf{x} into the vector relationship to determine the constraints on λ, μ and ν for the relationship to be satisfied.

♣ Example #1

Q: Find the vectors \mathbf{x} that satisfy the equation $\mathbf{x} = \mathbf{x} \times \mathbf{a} + \mathbf{b}$.

A: Steps (1 & 2): Use \mathbf{a} and \mathbf{b} and their vector product $\mathbf{a} \times \mathbf{b}$ as basis vectors.

Step (3): $\mathbf{x} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{a} \times \mathbf{b}$.

Step (4): Bung this expression for \mathbf{x} into the equation!

$$\lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{a} \times \mathbf{b} = [\lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{a} \times \mathbf{b}] \times \mathbf{a} + \mathbf{b} \quad (2.35)$$

$$= \mathbf{0} + \mu(\mathbf{b} \times \mathbf{a}) + \nu(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} + \mathbf{b} \quad (2.36)$$

But using the expression for the vector triple product

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} = \mathbf{a} \times (\mathbf{b} \times \mathbf{a}) \quad (2.37)$$

$$= (\mathbf{a} \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a} = a^2\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a} \quad (2.38)$$

Hence

$$\lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{a} \times \mathbf{b} = -\nu(\mathbf{a} \cdot \mathbf{b})\mathbf{a} + (\nu a^2 + 1)\mathbf{b} - \mu(\mathbf{a} \times \mathbf{b}) \quad (2.39)$$

We have learned that any vector has a unique expression in terms of a basis set, so that the coefficients of \mathbf{a} , \mathbf{b} and $\mathbf{a} \times \mathbf{b}$ on either side of the equation must be equal.

$$\Rightarrow \lambda = -\nu(\mathbf{a} \cdot \mathbf{b}) \quad \mu = \nu a^2 + 1 \quad \nu = -\mu \quad (2.40)$$

so that

$$\mu = \frac{1}{1 + a^2} \quad \nu = -\frac{1}{1 + a^2} \quad \lambda = \frac{\mathbf{a} \cdot \mathbf{b}}{1 + a^2} . \quad (2.41)$$

So finally the solution in this case is a single point:

$$\mathbf{x} = \frac{1}{1 + a^2} ((\mathbf{a} \cdot \mathbf{b})\mathbf{a} + \mathbf{b} - (\mathbf{a} \times \mathbf{b})) \quad (2.42)$$

♣ Example #2

Q: The second example $\mathbf{x} \cdot \mathbf{a} = K$ is in 2A1A. Notice that the geometry is rather obvious in this case! \mathbf{x} must lie on the plane $\mathbf{x} \cdot \hat{\mathbf{a}} = K/a$. A plane with unit normal $\hat{\mathbf{a}}$ and perpendicular distance K/a from the origin. However, that does not answer the question.

A: Using the recipe you choose \mathbf{b} arbitrarily, and will find

$$\mathbf{x} = \lambda \mathbf{a} + \left[\frac{K - \lambda a^2}{\mathbf{a} \cdot \mathbf{b}} \right] \mathbf{b} + \nu(\mathbf{a} \times \mathbf{b}) \quad (2.43)$$

where λ and ν are free. This is correct, but why does it look so complicated? It is because \mathbf{b} has been chosen arbitrarily and is one of the basis vectors.

As we can see upfront that this must be a plane, a cunning selection would be to choose the second vector to be perpendicular to both \mathbf{a} and $\mathbf{a} \times \mathbf{b}$. You still have to choose \mathbf{b} arbitrarily, but using our knowledge, we can write **without further thought**

$$\mathbf{x} = \frac{K}{a^2} \mathbf{a} + \mu(\mathbf{a} \times (\mathbf{a} \times \mathbf{b})) + \nu(\mathbf{a} \times \mathbf{b}) . \quad \mu, \nu \text{ are free} \quad (2.44)$$

Can you see why?