Classical Mechanics

Lagrangian Mechanics

- ☞ Variational Principle ✓
- ☞ Lagrange Equations for an Unconstrained Particle
- ☞ Lagrange Equations for Several Unconstrained Particles
- ☞ Constrained Systems
- ☞ Holonomic Systems

Lagrange's Equations for Unconstrained Motion

Consider a particle moving unconstrained in 3 dimensions

The particle's kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{r}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and its potential energy

$$U = U(r) = U(x, y, z)$$

The Lagrangian is defined as

$$\mathfrak{L} = T - U \Rightarrow \mathfrak{L} = \mathfrak{L}(x,\ y,\ z,\ \dot{x},\ \dot{y},\ \dot{z})$$

$$\frac{\partial \mathfrak{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x \qquad \frac{\partial \mathfrak{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = p_x$$

Straightforward calculation shows that Newton's Law ☚ $\vec{F} = \dot{\vec{p}}$

$$\Downarrow$$

implies 3 Lagrangian equations in Cartesian coordinates

$$\frac{\partial \mathfrak{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{x}} \qquad \frac{\partial \mathfrak{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{y}} \qquad \frac{\partial \mathfrak{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{z}}$$

Hamilton's Principle

Euler-Lagrange equations then imply

$\Downarrow$

Hamilton's Principle
(1805 -1865)

The actual path which a particle follows between points 1 and 2 in a given time interval t_1 to t_2 is such that the action integral

$$S = \int_{t_1}^{t_2} \mathfrak{L}\, dt$$

is stationary when taken along the actual path

- **A particle's path is determined by Newton's second law $\vec{F} = m\vec{a}$**
- **The path is determined by 3 Lagrange equations**
  (at least in Cartesian coordinates)
- **The path is determined by Hamilton's principle**

Generalized Coordinates

Instead of Cartesian coordinates ☛ consider now

- spherical polar coordinates $(r,\ \theta,\ \phi)$
- cylindrical coordinates $(\rho,\ \phi,\ z)$
- or any set of "generalized coordinates" $(q_1,\ q_2,\ q_3)$
  satisfying $q_i = q_i(\vec{r})$ for $i = 1,\ 2,\ 3$ and $r = r(q_1,\ q_2,\ q_3)$

Next re-write the Lagrangian in terms of these new variables

$$\mathfrak{L} = \mathfrak{L}(q_1,\ q_2,\ q_3,\ \dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)$$

and the action integral

$$S = \int_{t_1}^{t_2} \mathfrak{L}(q_1,\ q_2,\ q_3,\ \dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)\, dt$$

The value of the integral is unaltered by this change of variables

$$\Downarrow$$

The statement that S is stationary
for variations of the path around the correct path
must still be true in the new coordinates

Generalized Coordinates (cont'd)

All in all ☛ the correct path must satisfy the 3 Euler-Lagrange equations

$$\frac{\partial \mathfrak{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}_1} \qquad \frac{\partial \mathfrak{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}_2} \qquad \frac{\partial \mathfrak{L}}{\partial q_3} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}_3}$$

Since these new coordinates are any set of generalized coordinates
the comment ☞ (at least in Cartesian coordinates)
can be omitted from our previous statement
Crucial step in deriving Lagrange's equations of motion

⇓

observation that yield a result equivalent to Newton's second law
true only if the original frame in which we wrote down $\mathfrak{L} = T - U$ is inertial
Though Lagrange's equations are true for any choice of $(q_1,\ q_2\, q_3)$
that may in fact be the coordinates of a non-inertial frame
must be careful when we first write down the Lagrangian

⇓

do so in an inertial frame

Motion in a Central Potential

Consider a particle of mass m moving in 2D in the central potential $U(r)$
this is clearly a two degree of freedom dynamical system
The particle's instantaneous position is most conveniently specified

$$\Downarrow$$

plane polar coordinates r and θ

these are our two generalized coordinates
The square of the particle's velocity can be written as

$$v^2 = \dot{r}^{\,2} + (r\,\dot{\theta})^2$$

the Lagrangian of the system takes the form

$$\mathfrak{L} = \frac{1}{2}\,m\,(\dot{r}^{\,2} + r^2\,\dot{\theta}^{\,2}) - U(r)$$

$$\frac{\partial \mathfrak{L}}{\partial \dot{r}} = m\,\dot{r} \qquad \frac{\partial \mathfrak{L}}{\partial r} = m\,r\,\dot{\theta}^{\,2} - dU/dr$$

$$\frac{\partial \mathfrak{L}}{\partial \dot{\theta}} = m\,r^2\,\dot{\theta} \qquad \frac{\partial \mathfrak{L}}{\partial \theta} = 0$$

Motion in a Central Potential (cont'd)

Equations of motion ☞ Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{r}}\right) - \frac{\partial \mathfrak{L}}{\partial r} = 0 \qquad \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathfrak{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}(m\,\dot{r}) - m\,r\,\dot{\theta}^2 + \frac{dU}{dr} = 0$$

$$\frac{d}{dt}\left(m\,r^2\,\dot{\theta}\right) = 0$$

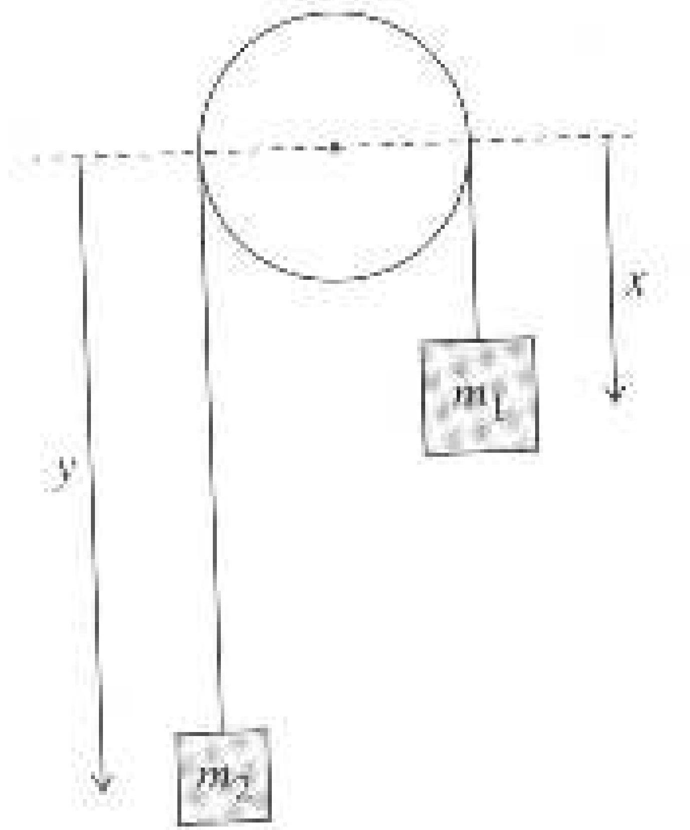
$$\ddot{r} - r\,\dot{\theta}^2 = -\frac{dV}{dr}$$

$$r^2\,\dot{\theta} = h$$

☞ $V = U/m$
☞ $h =$ constant

Atwood Machines

An Atwood machine consists of two weights of mass m_1 and m_2 connected by a light inextensible cord of length l which passes over a pulley



This is a one degree of freedom system whose instantaneous configuration is specified by the coordinate x

Atwood Machines (cont'd)

The kinetic energy of the system is

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2$$

The potential energy of the system takes the form

$$U = -m_1 \, g \, x - m_2 \, g \, (l - x)$$

It follows that the Lagrangian is written as

$$\mathfrak{L} = \frac{1}{2} (m_1 + m_2) \, \dot{x}^2 + g \, (m_1 - m_2) \, x + \text{constant}$$

The equation of motion is

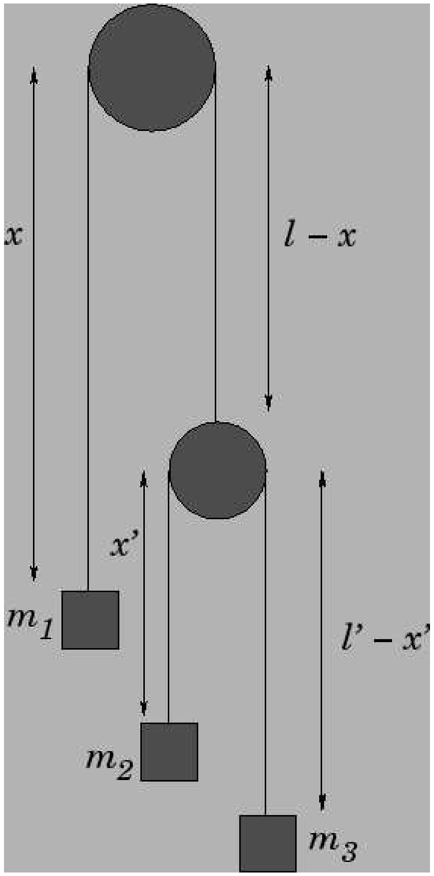
$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}}\right) - \frac{\partial \mathfrak{L}}{\partial x} = 0$$

yielding

$$(m_1 + m_2) \, \ddot{x} - g \, (m_1 - m_2) = 0 \Rightarrow \ddot{x} = \frac{g \, (m_1 - m_2)}{m_1 + m_2}$$

Atwood Machines (cont'd)

Atwood machine with one weight replaced by a second Atwood machine



The system now has two degrees of freedom
its instantaneous position is specified by the two coordinates x and x'

Atwood Machines (cont'd)

The kinetic energy of the system is

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (-\dot{x} + \dot{x}')^2 + \frac{1}{2} m_3 (-\dot{x} - \dot{x}')^2$$

The potential energy takes the form

$$U = -m_1 g x - m_2 g (l - x + x') - m_3 g (l - x + l' - x')$$

It follows that the Lagrangian of the system is

$$\begin{aligned} \mathfrak{L} &= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (-\dot{x} + \dot{x}')^2 + \frac{1}{2} m_3 (-\dot{x} - \dot{x}')^2 \\ &+ g (m_1 - m_2 - m_3) x + g (m_2 - m_3) x' + \text{constant} \end{aligned}$$

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}}\right) - \frac{\partial \mathfrak{L}}{\partial x} = 0 \qquad \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}'}\right) - \frac{\partial \mathfrak{L}}{\partial x'} = 0$$

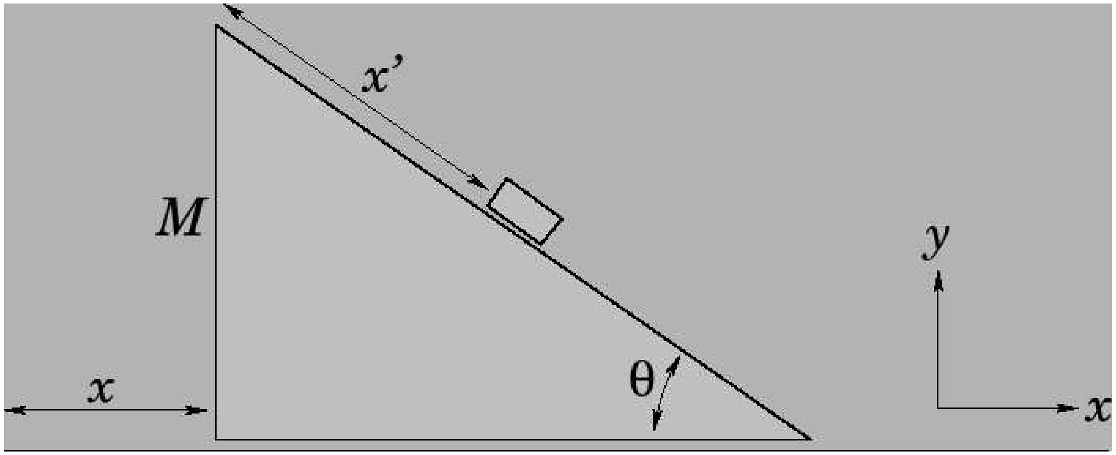
yielding

$$m_1 \ddot{x} + m_2 (\ddot{x} - \ddot{x}') + m_3 (\ddot{x} + \ddot{x}') - g (m_1 - m_2 - m_3) = 0$$

$$m_2 (-\ddot{x} + \ddot{x}') + m_3 (\ddot{x} + \ddot{x}') - g (m_2 - m_3) = 0$$

Sliding down a sliding plane

Consider the case of a particle of mass m sliding down a smooth inclined plane of mass M which is free to slide on a smooth horizontal surface



This is a two degree of freedom system

$\Downarrow$

need two coordinates to specify the configuration

CHOOSE

x ☚ horizontal distance of the plane from some reference point

x' ☚ particle's parallel displacement from some reference point on the plane

Define x- and y-axes as shown in the diagram

Sliding down a sliding plane (cont'd)

x- and y-components of the particle's velocity are

$$v_x = \dot{x} + \dot{x}' \cos\theta$$

$$v_y = -\dot{x}' \sin\theta$$

θ ☛ angle of inclination of the plane with respect to horizontal

$$\Downarrow$$

$$v^2 = v_x^2 + v_y^2 = \dot{x}^{\,2} + 2\,\dot{x}\,\dot{x}' \cos\theta + \dot{x}'^{\,2}$$

The kinetic energy of the system takes the form

$$T = \frac{1}{2} M\,\dot{x}^{\,2} + \frac{1}{2} m\,(\dot{x}^{\,2} + 2\,\dot{x}\,\dot{x}' \cos\theta + \dot{x}'^{\,2})$$

The potential energy is given by

$$U = -m\,g\,x' \sin\theta + \text{constant}$$

$$\Downarrow$$

It follows that the Lagrangian is written as

$$\mathfrak{L} = \frac{1}{2} M\,\dot{x}^{\,2} + \frac{1}{2} m\,(\dot{x}\cos\theta + \dot{x}'^{\,2}) + m\,g\,x' \sin\theta + \text{constant}$$

Sliding down a sliding plane (cont'd)

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}}\right) - \frac{\partial \mathfrak{L}}{\partial x} = 0 \qquad \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}'}\right) - \frac{\partial \mathfrak{L}}{\partial x'} = 0$$

$$\Downarrow$$

$$M\,\ddot{x} + m\,(\ddot{x} + \ddot{x}'\,\cos\theta) = 0$$

$$m\,(\ddot{x}' + \ddot{x}\,\cos\theta) - m\,g\,\sin\theta = 0$$

Solving for $\ddot{x}$ and $\ddot{x}'$

$$\ddot{x} = -\frac{g\,\sin\theta\,\cos\theta}{(m+M)/m - \cos^2\theta}$$

$$\ddot{x}' = \frac{g\,\sin\theta}{1 - m\,\cos^2\theta/(m+M)}$$

Lagrange Equations for Several Unconstrained Particles

Consider 2 particles moving unconstrained in 3 dimensions
the kinetic energy of the system is

$$T = \frac{1}{2} m_1 \dot{r_1}^2 + \frac{1}{2} m_2 \dot{r_2}^2$$

and its potential energy

$$U = U(r_1,\ r_2)$$

The Lagrangian is defined as

$$\mathfrak{L} = T - U$$

$$\Downarrow$$

$$\mathfrak{L} = \mathfrak{L}(r_1,\ r_2,\ \dot{r}_1,\ \dot{r}_2) = \frac{1}{2} m_1 \dot{r_1}^2 + \frac{1}{2} m_2 \dot{r_2}^2 - U(r_1,\ r_2)$$

As usual the forces on the two particles are

☛ $\vec{F}_1 = -\vec{\nabla}_1 U$

☛ $\vec{F}_2 = -\vec{\nabla}_2 U$

Newton's second law can be applied to each particle and yields 6 equations

$$F_{1x} = \dot{p}_{1x} \quad F_{1y} = \dot{p}_{1y} \quad F_{1z} = \dot{p}_{1z} \quad F_{2x} = \dot{p}_{2x} \quad F_{2y} = \dot{p}_{2y} \quad F_{2z} = \dot{p}_{2z}$$

Lagrange Equations for Several Unconstrained Particles (cont'd)

Each equation is equivalent to a corresponding Lagrange equation

$$\frac{\partial \mathfrak{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{x}_1} \qquad \frac{\partial \mathfrak{L}}{\partial y_1} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{y}_1} \qquad \frac{\partial \mathfrak{L}}{\partial z_1} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{z}_1}$$

$$\frac{\partial \mathfrak{L}}{\partial x_2} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{x}_2} \qquad \frac{\partial \mathfrak{L}}{\partial y_2} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{y}_2} \qquad \frac{\partial \mathfrak{L}}{\partial z_2} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{z}_2}$$

The 6 equations imply that $S = \int_{t_1}^{t_2} \mathfrak{L}\, dt$ is stationary

Changing to any set of generalized coordinates we have

$$\frac{\partial \mathfrak{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}_1} \qquad \frac{\partial \mathfrak{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}_2} \qquad \frac{\partial \mathfrak{L}}{\partial q_3} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}_3}$$

$$\frac{\partial \mathfrak{L}}{\partial q_4} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}_4} \qquad \frac{\partial \mathfrak{L}}{\partial q_5} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}_5} \qquad \frac{\partial \mathfrak{L}}{\partial q_6} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}_6}$$

Lagrange Equations for Several Unconstrained Particles (cont'd)

HOMEWORK

Consider a system of N unconstrained particles
(e.g. gas of N molecules)

Show that there are $3N$ Lagrange equations

$$\frac{\partial \mathfrak{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}_i} \qquad i = 1, 2, \ldots, 3N$$

valid for any choice of the $3N$ coordinates $q_1, \ q_2, \ q_3, \ \ldots q_{3N}$ needed to describe the motion of the N particles

Constrained Systems

Consider an arbitrary system of N particles $\alpha = 1, \ldots N$ with position r_α

the parameters $q_1 \ldots q_n$ are a set of generalized coordinates for the system

$$\Updownarrow$$

I ✉ each position r_α can be expressed as a function of $q_1 \ldots, \; q_n$ and possibly t

$$r_\alpha = r_\alpha(q_1, \ldots, \; q_n, \; t) \qquad [\alpha = 1, \ldots, \; N]$$

II ✉ each q_i can be expressed in terms of the r_α and possibly t

$$q_i = q_i(r_1, \ldots, \; r_N, \; t) \qquad [i = 1, \ldots, \; n]$$

III ✉ The number of generalized coordinates (n) is the smallest number that allows the system to be parametrized in this way

In a 3-dimensional world

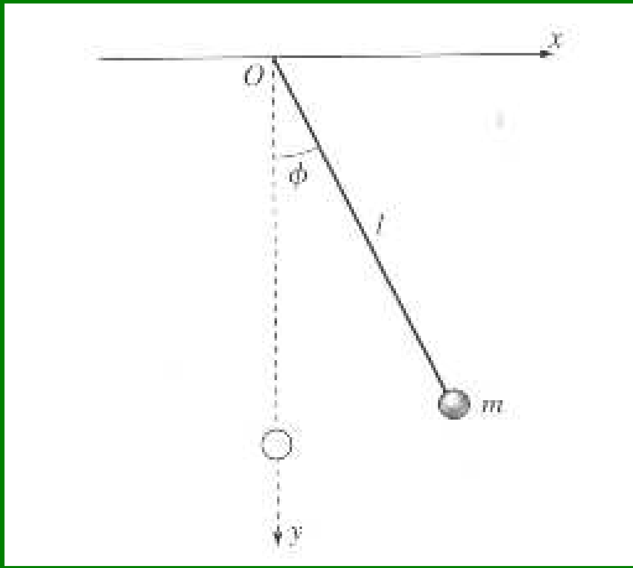
$$\Downarrow$$

the number of n-generalized coordinates for N particles
is certainly no more than $3N$

For a constrained system is usually less ☞ sometimes dramatically so!

The Simple Pendulum

Consider a bob of mass m fixed to a massless rod
(which is pivoted at O and free to swing without friction in the xy plane)



The bob moves in both the x and y directions
(but it is constrained by the rod)

$$\Downarrow$$

$$\sqrt{x^2 + y^2} = l \text{ remains constant}$$

$$\Downarrow$$

only 1 coordinate is independent ☞ the system has 1 degree of freedom

The Simple Pendulum (cont'd)

Need to eliminate on of the coordinates

one obvious possibility $y = \sqrt{l^2 - x^2}$

Simpler solution ☞ express both x and y in terms of ϕ
($\phi \equiv$ angle between the pendulum and its equilibrium position)

The kinetic energy is

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot{\phi}^2$$

The potential energy is

$$U = m g h$$

h **☞ height of the bob above its equilibrium position**

$$\Downarrow$$

$$h = l(1 - cos\phi)$$

The Lagrangian is

$$\mathfrak{L} = T - U = \frac{1}{2} m l^2 \dot{\phi}^2 - m g l (1 - \cos\phi)$$

The Simple Pendulum (cont'd)

Equation of Motion

$$\frac{\partial \mathfrak{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{\phi}}$$

$$-m\,g\,l\;\sin\phi = \frac{d}{dt}(ml^2\dot{\phi}) = m\,l^2\,\ddot{\phi}$$

✐ $-mgl\sin\phi = \tau$ ☞ torque exerted by gravity on the pendulum

➠ $ml^2 = I$ ☞ pendulum's moment of inertia

✎ $\ddot{\phi} = \alpha$ ☞ angular acceleration

Lagrange equation for the simple pendulum reproduces the familiar result

$$\Downarrow$$

$$\tau = I\alpha$$

The Spherical Pendulum

Consider a mass m at the end of light inextensible string of length l

Suppose that the mass is free to move in any direction
(as long as the string remains taut)

the fixed end of the string be located at the origin of coordinate system

We can define Cartesian coordinates ☞ (x, y, z)
such that the z-axis points vertically upward

We can also define spherical polar coordinates ☞ (r, θ, ϕ)
whose axis points along the $-z$-axis

The latter coordinates are the most convenient

$\Downarrow$

r is constrained to always take the value l

the two angular coordinates θ and ϕ are free to vary independently

$\Downarrow$

this is clearly a two degree of freedom system

The Spherical Pendulum (cont'd)

The Cartesian coordinates can be written in terms of angular coordinates

$$x = l \sin\theta \sin\phi$$

$$y = l \sin\theta \cos\phi$$

$$z = -l \cos\theta$$

The potential energy of the system is

$$U = m\,g\,z = -m\,g\,l \cos\theta$$

The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}ml^2\,(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$$

The Lagrangian of the system is written

$$\mathfrak{L} = \frac{1}{2}\,m\,l^2\,(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + m\,g\,l \cos\theta$$

The Spherical Pendulum (cont'd)

The Lagrangian is independent of the angular coordinate ϕ

⇓

$$p_\phi = \frac{\partial \mathfrak{L}}{\partial \dot{\phi}} = m\, l^2 \sin^2\theta\, \dot{\phi}$$

⇓

p_ϕ ☞ the angular momentum of the system about the z-axis

⇓

is a constant of the motion

Note that neither the tension in the string nor the force of gravity exert a torque about the z-axis

Conservation of angular momentum about the z-axis

⇓

$$\sin^2\theta\, \dot{\phi} = h = \text{constant}$$

The Spherical Pendulum (cont'd)

The equation of motion of the system

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathfrak{L}}{\partial \theta} = 0$$

is

$$\ddot{\theta} + \frac{g}{l}\,\sin\theta - \sin\theta\,\cos\theta\,\dot{\phi}^2 = 0$$

or equivalently

$$\ddot{\theta} + \frac{g}{l}\,\sin\theta - h^2\,\frac{\cos\theta}{\sin^3\theta} = 0$$

Suppose that $\phi = \phi_0 = \text{constant}$

$$\Downarrow$$

$$\dot{\phi} = h = 0$$

$$\Downarrow$$

$$\ddot{\theta} + \frac{g}{l}\,\sin\theta = 0$$

simple pendulum whose motion is restricted to the vertical plane $\phi = \phi_0$

The Spherical Pendulum (cont'd)

Suppose that $\theta = \theta_0 = \text{constant}$

$$\Downarrow$$

$$\dot{\phi} = \dot{\phi}_0 = \text{constant}$$

$$\Downarrow$$

the pendulum bob rotates uniformly in a horizontal plane

$$\dot{\phi}_0 = \sqrt{\frac{g}{d}}$$

$d = l\,\cos\theta_0$ ☞ vertical distance of the plane of rotation below the pivot point

☞ This type of pendulum is usually called a conical pendulum
string attached to pendulum bob sweeps out a cone as the bob rotates

The Spherical Pendulum (cont'd)

Suppose that the motion is almost conical

the value of θ remains close to the value θ_0 ☞ $\theta = \theta_0 + \delta\theta$

Taylor expanding the eqaution of motion to first order in $\delta\theta$

the zeroth order terms cancel out ☞ we are left with

$$\delta\ddot{\theta} + \dot{\phi}_0^2\,(1 + 3\,\cos^2\theta_0)\,\delta\theta \simeq 0$$

Solution to this equation

$$\theta \simeq \theta_0 + \delta\theta_0\,\cos(\Omega\,t) \qquad \text{with} \qquad \Omega = \dot{\phi}_0\,\sqrt{1 + 3\,\cos^2\theta_0}$$

☛ The angle θ executes simple harmonic motion about its mean value θ_0 (at the angular frequency Ω)

☛ The azimuthal angle ϕ increases by

$$\Delta\phi \simeq \dot{\phi}_0\,\frac{\pi}{\Omega} = \frac{\pi}{\sqrt{1 + 3\,\cos^2\theta_0}}$$

as the angle of inclination to the vertical θ
goes between successive maxima and minima

The Spherical Pendulum (cont'd)

- Suppose that θ_0 is small ☞ $\Delta\phi$ is slightly greater than $\pi/2$

If $\Delta\phi$ were exactly $\pi/2$

$$\Downarrow$$

the pendulum bob would trace out the outline of a slightly warped circle
(something like the outline of a potato chip or a saddle)
The fact that $\Delta\phi$ is slightly greater than $\pi/2$ means that this shape precesses
about the z-axis in the same direction as the direction rotation of the bob
The precession rate increases as the angle of inclination θ_0 increases

- Suppose that θ_0 is slightly less than $\pi/2$ ☞ $\Delta\phi$ is slighly less than π
(of course θ_0 can never exceed $\pi/2$)

If $\Delta\phi$ were exactly π

$$\Downarrow$$

the pendulum bob would trace out the outline of a slightly tilted circle
The fact that $\Delta\phi$ is slightly less than π means that this shape precesses
about the z-axis in the opposite direction to the direction of rotation of the bob
The precession rate increases as the angle of inclination θ_0 decreases below $\pi/2$

Holonomic Systems

➠ **Number of degrees of freedom of a system**

- number of coordinates that can be independently varied in a small displacement
- number of independent directions in which the system can move from any given initial configuration

When the number of an N particle system in 3 dimensions is less than 3N

$\Downarrow$

the system is constrained

In 2 dimensions the corresponding number is $2N$

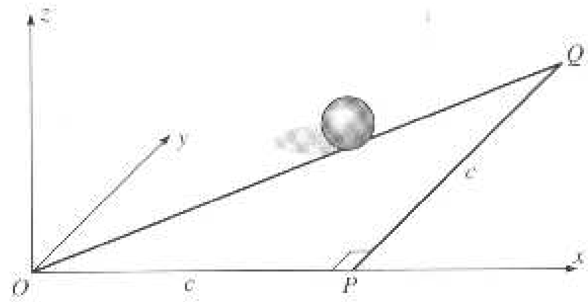
$\Downarrow$

the bob a simple pendulum with 1 degree of freedom is constrained

Holonomic Systems ☞ Systems that have n degrees of freedom and can be described by n generalized coordinates

but then ... aren't all systems holonomic?

Holonomic Systems (cont'd)



Consider a rubber ball that is free to roll on a horizontal table
(but not to slide nor to spin about a verical axis)

Starting at any position $(x,\ y)$ it can move only in two independent directions
☞ the ball has 2 degrees of freedom

Can this configuration be described by two coordinates $x,\ y$ of its center?

- Place the ball at the origin O and make a mark on its top
- Roll the ball along the x-axis for a distance equal to a circumference c to a point P where the mark will be once again at the top
- Roll it the same distance c along y-axis to Q ☞ the mark is again on top
- Roll it straight back to the origin along the hypothenuse of the triangle OPQ since the last move has length $\sqrt{2}c$ brings the ball back to its starting point but with the mark no longer on the top

The position x, y has returned to its initial value
but the ball has now a different orientation!!!

Holonomic Systems (cont'd)

Evidently the 2 coordinates are not enough to specify a unique configuration the ball has 2 degrees of freedom but more than 2 generalized coordinates

It is a non-holonomic system!

For any holonomic system with:

- **generalized coordinates** $q_1, \ldots, q_n$
- **potential energy** $U(q_1, \ldots, q_n, t)$

the evolution in time is determined by n **Lagrange equations**

$$\frac{\partial \mathfrak{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}_i} \qquad [i = 1, \ldots, n]$$

with $\mathfrak{L} = T - U$