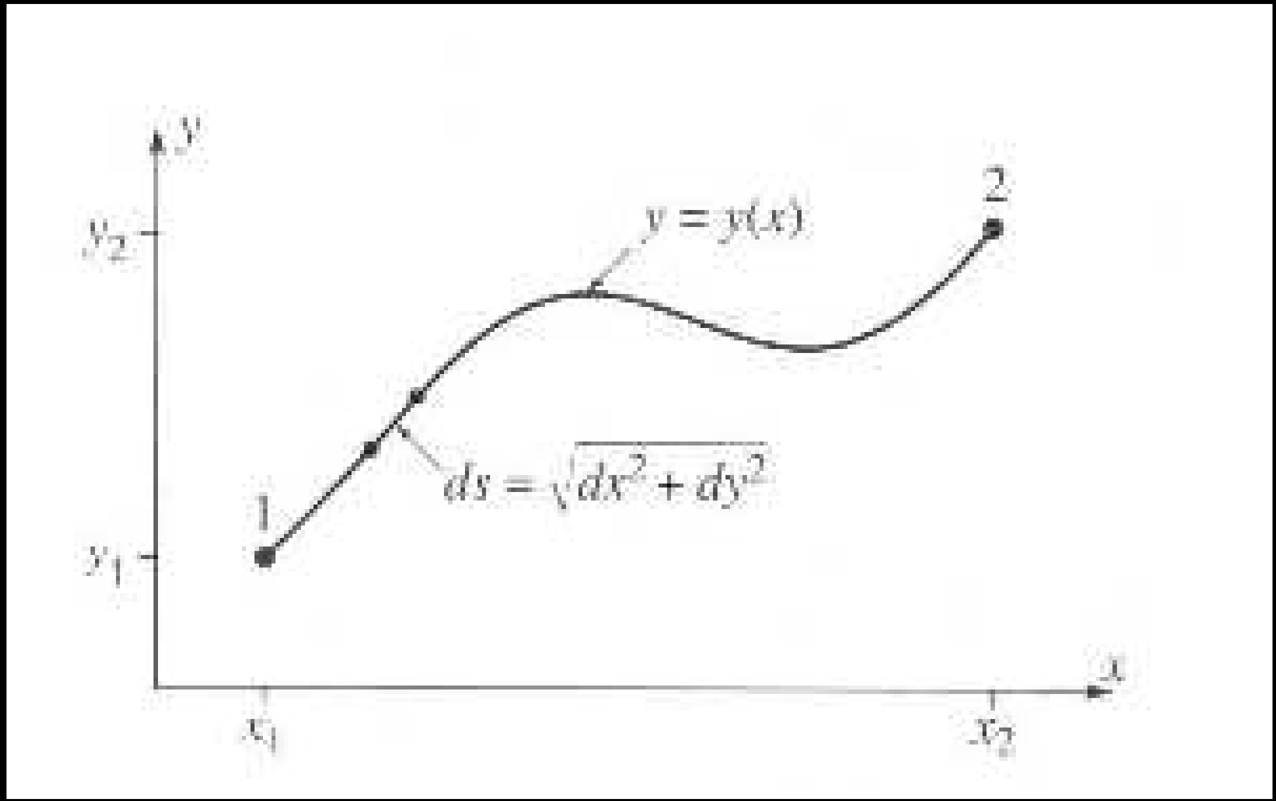
Classical Mechanics

Lagrangian Mechanics

☞ **Variational Principle**

The Shortest Path between Two Points

Given two points in a plane



what is the shortest path between them?

The Shortest Path between Two Points (cont'd)

The length of a short segment of the path is

$$ds = \sqrt{dx^2 + dy^2}$$

$$\Downarrow$$

$$dy = \frac{dy}{dx} dx \equiv y'(x) dx$$

$$\Downarrow$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + [y'(x)]^2}\, dx$$

The total length of the path between points 1 and 2 is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + [y'(x)]^2}\, dx$$

This equation puts our problem in a mathematical form

$$\Downarrow$$

find the function $y(x)$ for which the integral is minimum

Fermat's Priciple

What is the path that light follows between two points?

⇓

Fermat (1601 - 1665)

⇓

the path for which the time of travel of the light is minimum

The time for light to travel a short distance ds is ds/v

$v \equiv c/n$ speed of light in a medium with refractive index n

$$\text{time of travel} = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \frac{1}{c}\int_1^2 n\, ds$$

In general ☞ refractive index can vary

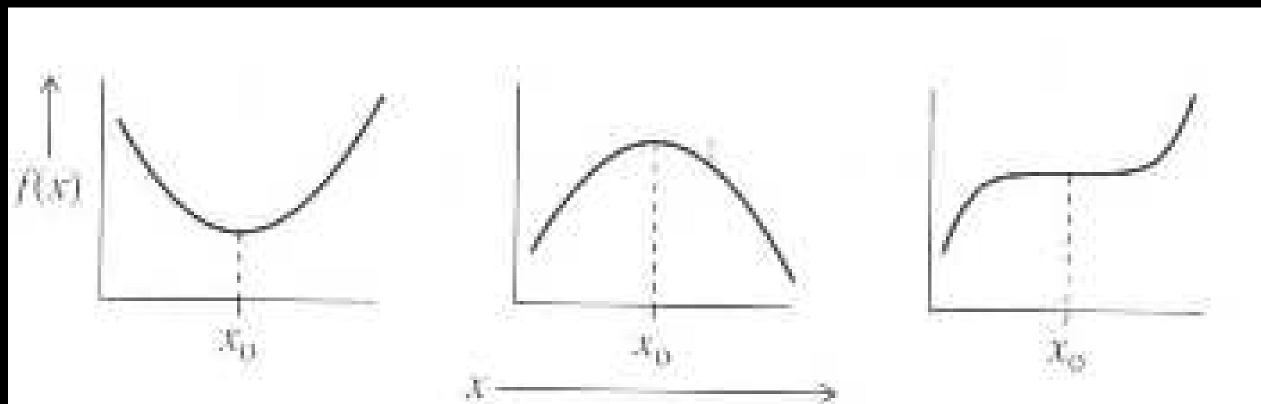
$$\int_1^2 n(x,y) ds = \int_{x_1}^{x_2} n(x,y)\sqrt{1+[y'(x)]^2}\, dy$$

Calculus of Variations

Standard minimization problem of elementary calculus

unknown value of the variable x at which a known function f(x) has a minimum

- **Recall that if $df/dx = 0$ at x_0 there are three possibilities**



- If $d^2f/dx^2 > 0 \Rightarrow f$ has a minimum
- If $d^2f/dx^2 < 0 \Rightarrow f$ has a maximum
- If $d^2f/dx^2 = 0 \Rightarrow$ there may be a minimum, a maximum, or neither

New problem ☞ one step more complicated

Calculus of Variations

how infinitesimal variations of a path change an integral

The Euler-Lagrange Equation

Consider an integral of the form

$$S = \int_{x_1}^{x_2} f[y(x),\, y'(x),\, x]\, dx$$

$y(x)$ ☞ unknown curve joining points (x_1, y_1) and (x_2, y_2)

$y(x_1) = y_1$ $\qquad$ $y(x_2) = y_2$

We have to find the curve that makes S a minimum

f ☞ function of 3 variables $f = f(y, y', x)$

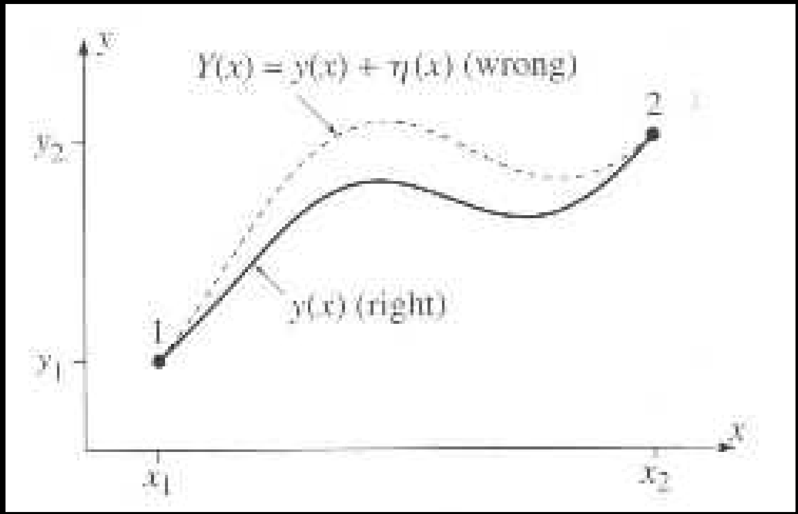
$\Downarrow$

but integral follows path $y = y(x)$

$\Downarrow$

integrand $f[y(x), y'(x), x]$ is actually a function of just one variable x

The Euler-Lagrange Equation (cont'd)



If $y(x)$ ☞ right solution

⇓

S evaluated for $y(x)$ is less than for any neighborhood curve $Y(x)$

convinient to write

$$Y(x) = y(x) + \eta(x)$$

since $Y(x)$ must pass through points 1 and 2

⇓

$$\eta(x_1) = \eta(x_2) = 0$$

The Euler-Lagrange Equation (cont'd)

The integral taken along the wrong curve $Y(x)$
must be larger than that along the right curve $y(x)$
no matter how close is the former to the latter

to express this requirement ☞ introduce parameter α

$$\Downarrow$$

$$Y(x) = y(x) + \alpha\ \eta(x)$$

The integral S taken along the curve $Y(x)$ now depends on α

$$\Downarrow$$

$$S(\alpha)$$

The right curve $y(x)$ is obtained by setting $\alpha = 0$

$$\Downarrow$$

reduction to traditional problem from elementary calculus

$$\Downarrow$$

$dS/d\alpha = 0$ when $\alpha = 0$

The Euler-Lagrange Equation (cont'd)

$$\begin{aligned} S(\alpha) &= \int_{x_1}^{x_2} f(Y,\ Y',\ x)\,dx \\ &= \int_{x_1}^{x_2} f(y+\alpha\eta,\ y'+\alpha\eta',\ x)\,dx \end{aligned}$$

differentiate with respect to α

$$\Downarrow$$

$$\frac{\partial f(y+\alpha\eta,\ y'+\alpha\eta',\ x)}{\partial\alpha} = \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}$$

$$\Downarrow$$

$$\begin{aligned} \frac{dS}{d\alpha} &= \int_{x_1}^{x_2} \frac{\partial f}{\partial\alpha}\,dx \\ &= \int_{x_1}^{x_2} \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\,dx \\ &= 0 \end{aligned}$$

The Euler-Lagrange Equation (cont'd)

Re-write second term on the right using integration by parts

$$\int_{x_1}^{x_2} \eta'(x) \frac{\partial f}{\partial y'}\, dx = \eta(x) \frac{\partial f}{\partial y'}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

endpoint term is zero

$$\Downarrow$$

$$\int_{x_1}^{x_2} \eta'(x) \frac{\partial f}{\partial y'}\, dx = -\int_{x_1}^{x_2} \eta(x)\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

$$\Downarrow$$

$$\int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) dx = 0$$

This condition must be satisfied for any choice of the function $\eta(x)$

We can conclude that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\forall x \in x_1 \leq x \leq x_2$$

if all the functions concerned are continuous

Leonhard Euler (1707-1783) Joseph Lagrange (1736-1813)

The Shortest Path between Two Points (cont'd)

We saw that the length of a path between points 1 and 2 is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

that has the standard form

$$f(y, y', x) = \sqrt{1 + y'^2}$$

$$\frac{\partial f}{\partial y} = 0 \qquad\qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

$$\Downarrow$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \Rightarrow \frac{\partial f}{\partial y'} = C$$

$$\Downarrow$$

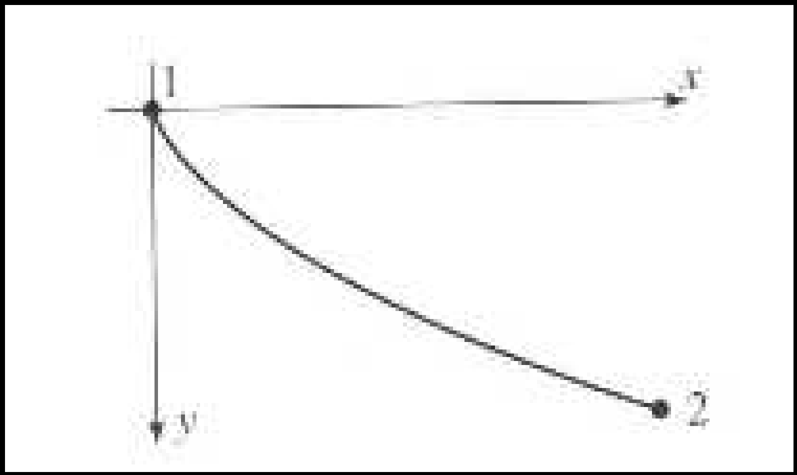
$$y'^2 = C^2(1 + y'^2) \Rightarrow y'^2 = \widetilde{C} \Rightarrow y'(x) = m$$

integration leads to

$$y(x) = mx + b$$

Brachistochrone

shape of the track on which
particle released from point 1 will reach point 2 in the minimum possible time



brachistos (shorter) chronos (time)

$$\text{time}(1 \rightarrow 2) = \int_1^2 \frac{ds}{v}$$

speed at any height y is determined by conservation of energy

$$\Downarrow$$

$$v = \sqrt{2gy}$$

Brachistochrone (cont'd)

Take y as independent variable → unknown path $x = x(y)$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'^2(y) + 1}\; dy$$

$$\text{time}(1 \to 2) = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{[x'(y)]^2 + 1}}{\sqrt{y}}\; dy$$

To find the path that makes the time as small as possible

☛ use Euler-Lagrange

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'}$$

$$\Downarrow$$

$$\frac{x'^2}{y(1 + x'^2)} = \text{constant} = \frac{1}{2a}$$

$$\Downarrow$$

$$x' = \sqrt{\frac{y}{2a - y}}$$

Brachistochrone (cont'd)

$$x = \int \sqrt{\frac{y}{2a - y}}\, dy$$

use unlikely looking substitution $y = a(1 - \cos\theta)$

$$\begin{aligned} x &= a \int (1 - \cos\theta) d\theta \\ &= a(\theta - \sin\theta) + \mathcal{C} \end{aligned}$$

We have chosen the initial point 1 to have $x = y = 0$

$$\Downarrow$$

initial $\theta = 0 \to \mathcal{C} = 0$

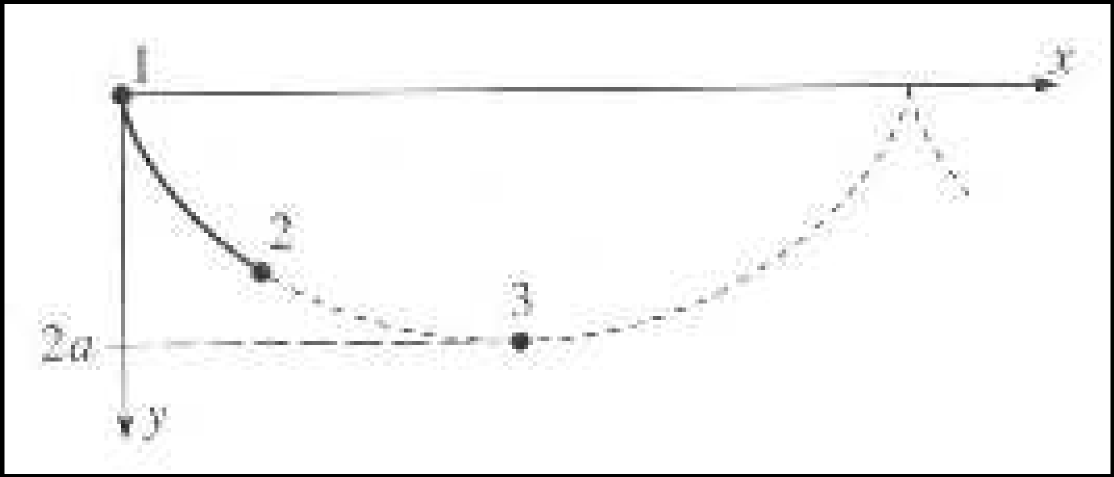
final parametric equation for the path is

$$x = a(\theta - \sin\theta) \qquad\qquad y = a(1 - \cos\theta)$$

with the constant a chosen so that the curve passes through $(x_2,\ y_2)$

Brachistochrone (cont'd)

In this figure we have continued the curve with dashes beyond the point 2
☛ **curve that solves the brachistochrone problem happens to be a cycloid**



the curve trace out by a point on a rim of a wheel of radius a
rolling along the underside of the x-axis

If we release a cart from rest at point 2 and let it roll to the bottom of the curve

the time to roll 2 to 3 is the same whatever the position of 2
anywhere between 1 and 3

oscillations of a cart rolling back and forth a cycloid-shape track are *isochronous*!
(period perfectly independent of the amplitude)

Brachistochrone (cont'd)

From the parametric equation we obtain the derivatives

$$x' = a(1 - \cos\theta) \qquad\qquad y' = a\sin\theta$$

the element of path length is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'^2 + y'^2}\, d\theta = a\sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta$$

$$\Downarrow$$

$$ds = a\sqrt{2(1-\cos\theta)}\, d\theta$$

the speed of the cart is given by conservation of energy as

$$v = \sqrt{2g(y_0 - y)} = \sqrt{2ga(\cos\theta_0 - \cos\theta)}$$

the required time is

$$t = \int_{P_0}^{P} \frac{ds}{v} = \int_{\theta_0}^{\pi} \frac{a\sqrt{2(1-\cos\theta)}}{\sqrt{2ga(\cos\theta_0 - \cos\theta)}} d\theta$$

Brachistochrone (cont'd)

Using the substitution $\theta = \pi - 2\alpha$ plus a couple of trig identities

$$t = 2\sqrt{\frac{a}{g}} \int_0^{\alpha_0} \frac{\cos\alpha}{\sqrt{\sin^2\alpha_0 - \sin^2\alpha}} d\alpha$$

Using the substitution $\sin\alpha = u$ and $u/u_0 = v$

$$t = 2\sqrt{\frac{a}{g}} \int_0^{u_0} \frac{du}{\sqrt{u_0^2 - u^2}} = 2\sqrt{\frac{a}{g}} \int_0^{1} \frac{dv}{\sqrt{1 - v^2}}$$

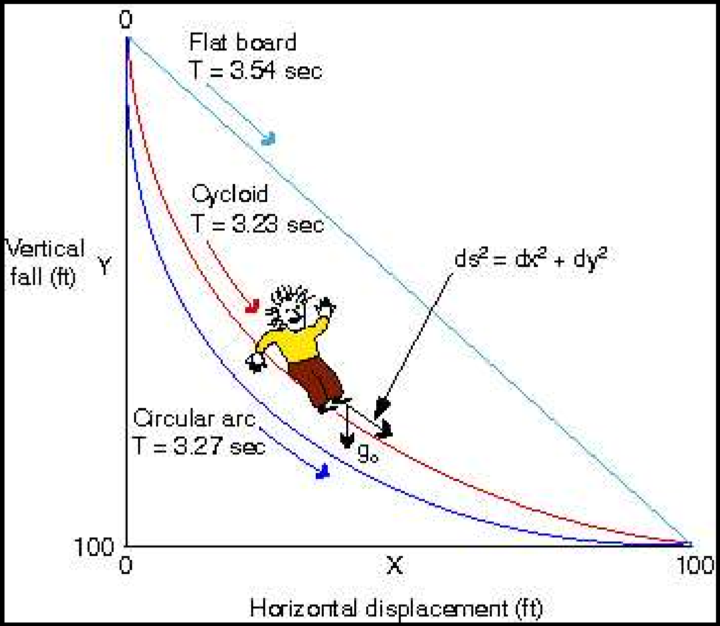
$$t = \pi\sqrt{\frac{a}{g}}$$

which is independent of θ_0!!!

The higher the starting point P_0
the further the car has to go
but the steeper the initial slope and the faster the car goes
On a cycloid these two effects perfectly cancel
the time to reach P is independent of the position of P_0

Brachistochrone (cont'd)

Homework: verify that



More than Two Variables

- there are several dependent variables

For most applications in mechanics

- though fortunately only one independent variable t

$$x = x(t) \quad y = y(t)$$

the length of a small segment of the path is

$$ds^2 = \sqrt{dx^2 + dy^2} = \sqrt{x'^2(t) + y'^2(t)}dt$$

$$\Downarrow$$

the total path length is

$$L = \int_{t_1}^{t_2} \sqrt{x'^2(t) + y'^2(t)}\, dt$$

More than Two Variables (cont'd)

$$S = \int_{t_1}^{t_2} f[x(t), y(t), x'(t), y'(t), t] dt$$

the "correct" path is given by

$$x = x(t) \quad y = y(t)$$

a neighboring "wrong" path is of the form

$$x = x(t) + \alpha\, \xi(t) \qquad y = y(t) + \beta\, \eta(t)$$

$$\frac{\partial S}{\partial \alpha} = 0 \qquad \frac{\partial S}{\partial \beta} = 0$$

for $\alpha = \beta = 0$

$$\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial f}{\partial x'} \qquad \frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial f}{\partial y'}$$