Classical Mechanics

Newtonian Gravity

☞ Newton's Law
☞ Poisson's Equation
☞ Ocean Tides

Newton's Law of Universal Gravitation

Gravity is the force that maintains planets in orbit around the sun

It was first correctly described by Isaac Newton in 1687

According to Newton ☞ any two point mass objects

(or spherically symmetric objects of finite extent)

exert a force of attraction on one another

This force
- ✐ points along the line of centers joining the objects
- ➠ is directly proportional to the product of the objects' masses
- ✎ is inversely proportional to the square of the distance between them

⇓

$$F = -G\frac{m\,M}{r^2}\,\vec{e}_r$$

A laboratory verification of the law and determination of the gravitational constant was made in 1798 by Henry Cavendish using a torsion balance with 2 small spheres fixed at the end of a light rod

$$G = 6.6726 \pm 0.0008 \times 10^{-11}\ \mathrm{N\ m^2/kg^{-2}}$$

Newton's Law of Universal Gravitation (cont'd)

Gravitational Law strictly applies to point particles

If one of the particles is replaced by a body with certain extension

$\Downarrow$

we need additional hypothesis to calculate the force

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assume the gravitational force field is a linear field

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net gravitational force on a particle due to many other particles

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vector sum of all individual forces

For a body consisting of a continuous distribution of matter

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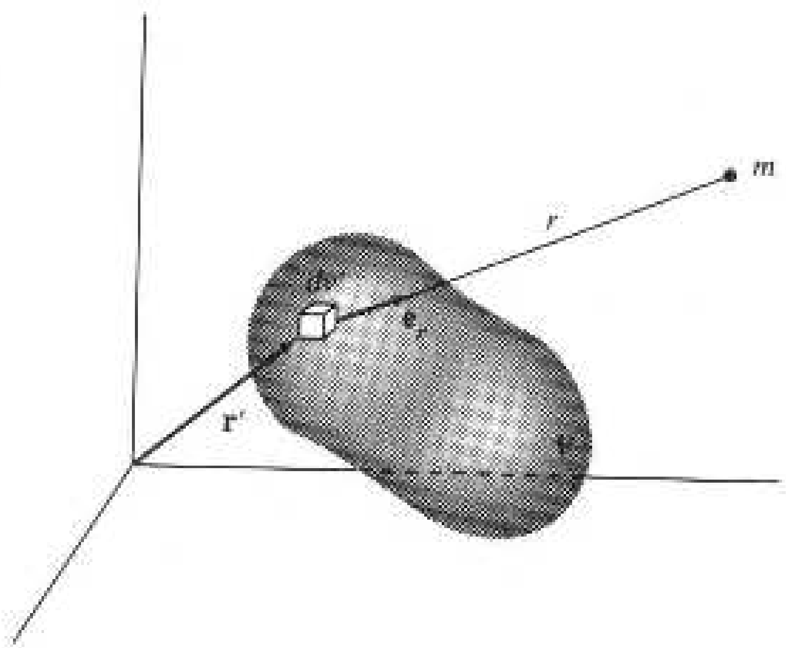
the sum becomes an integral

Newton's Law of Universal Gravitation(cont'd)



- $\rho(\vec{r}\,')$ $\equiv$ mass density
- dv' $\equiv$ element of volume at the position defined by $\vec{r}\,'$
  from arbitrary origin to the point within the mass distribution

$$F = -Gm \int_V \frac{\rho(\vec{r}\,')}{r^2}\, \vec{e}_r\, dv'$$

Gravitational Vector Field $\vec{g}$

Vector representing the force per unit mass excerted on a particle in the field of a body of mass M

$$\Downarrow$$

$$\vec{g} = \frac{\vec{F}}{m} = -G\frac{M}{r^2}\,\vec{e}_r$$

for a continuous distribution of matter

$$\Downarrow$$

$$\vec{g} = -G\int_V \frac{\rho(\vec{r}')}{r^2}dv'$$

The quantity $\vec{g}$ has the dimension of force per unit mass

Near the surface of the Earth $\vec{g}$ is the gravitational acceleration constant

$$|\vec{g}| \approx 9.8\mathrm{m/s}^2$$

Gravitational Potential

Gravitational Vector Field varies $\propto 1/r^2$

$\Downarrow$

satisfies the requirement that permits $\vec{g}$
to be represented as the gradient of a scalar function

$\Downarrow$

$$\vec{g} = -\vec{\nabla}\Phi$$

$\Phi \equiv$ gravitational potential

Since $\vec{g}$ has only radial variation

$\Downarrow$

can have at most variation with r

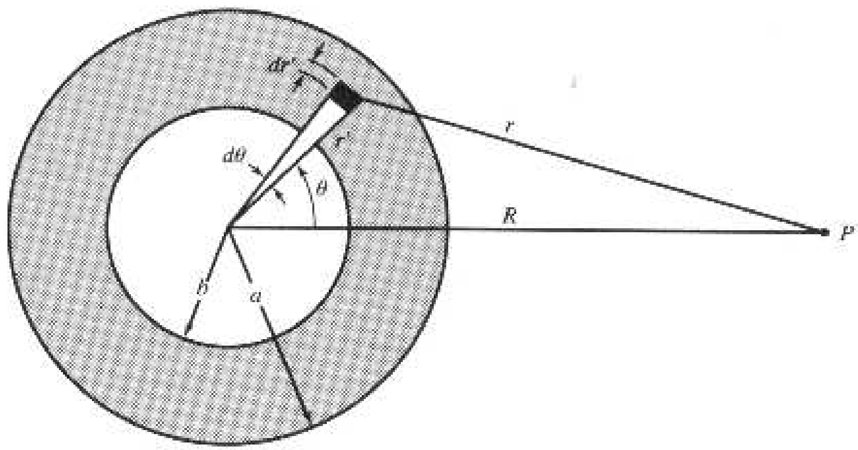
$$\vec{\nabla}\Phi = \frac{d\Phi}{dr}\vec{e}_r = G\,\frac{M}{r^2}\vec{e}_r$$

Integrating

$$\Phi = -G\,\frac{M}{r}$$

Gravitational Potential (cont'd)

Determine the gravitational potential inside and outside a spherical shell
of inner radius b and outer radius a ☞ homogeneous mass distribution $\rho(r') = \rho$



Symmetry about the line connecting the center of the sphere and the field point P

⇓

azimuthal angle immediately integrated in the expression for the potential

$$\begin{aligned}\Phi &= -G\int_V \frac{\rho(r')}{r} dv' \\ &= -2\pi\rho G \int_b^a r'^2 dr' \int_0^\pi \frac{\sin\theta}{r}\, d\theta\end{aligned}$$

Gravitational Potential (cont'd)

According to the law of cosines ☞ $r^2 = r'^2 + R^2 - 2r'R\cos\theta$

Since $R =$ constant ☞ for given r' we can differentiate this equation

$$2r dr = 2r'R\sin\theta d\theta \Rightarrow \frac{\sin\theta}{r} d\theta = \frac{dr}{r'R}$$

$$\Phi = -\frac{2\pi\rho G}{R}\int_b^a r' dr' \int_{r_{\rm min}}^{r_{\rm max}} dr$$

The limits on the integral over dr depend on the location of point P

I. P is outside the shell

$$\begin{aligned}\Phi(R > a) &= -\frac{2\pi\rho G}{R}\int_b^a r' dr' \int_{R-r'}^{R+r'} dr \\ &= -\frac{4\pi\rho G}{R}\int_b^a r'^2 dr' \\ &= -\frac{4}{3}\frac{\pi\rho G}{R}(a^3 - b^3)\end{aligned}$$

Mass of the shell ☞ $M = \frac{4}{3}\pi\rho(a^3 - b^3)$

$$\Downarrow$$

$$\Phi(R > a) = -\frac{GM}{R}$$

Gravitational Potential (cont'd)

$II.$ P is inside the shell

$$\begin{aligned} \Phi(R < b) &= -\frac{2\pi\rho G}{R}\int_b^a r' dr' \int_{r'-R}^{r'+R} dr \\ &= -4\pi\rho G \int_b^a r' dr' \\ &= -2\pi\rho G(a^2 - b^2) \end{aligned}$$

The potential is constant and independent of the position within the shell

$III.$ P is within the shell

Substitute by R the lower limit of integration in the expression $\Phi(R < b)$

Substitute by R the upper limit of integration in the expression $\Phi(R > a)$

$$\begin{aligned} \Phi(b < R < a) &= -\frac{4\pi\rho G}{3R}(R^3 - b^3) - 2\pi\rho G(a^2 - R^2) \\ &= -4\pi\rho G\left(\frac{a^2}{2} - \frac{b^3}{3R} - \frac{R^2}{6}\right) \end{aligned}$$

Gravitational Potential (cont'd)

SUMMARY

- The potential at any point outside spherically symmetric distribution of matter

$\Downarrow$

is independent of the size of the distribution

$\Downarrow$

to calculate external potential ☞ consider all the mass concentrated at the center

- The potential is constant inside any spherically symmetric mass shell

Magnitude of the vector field $\vec{g}$

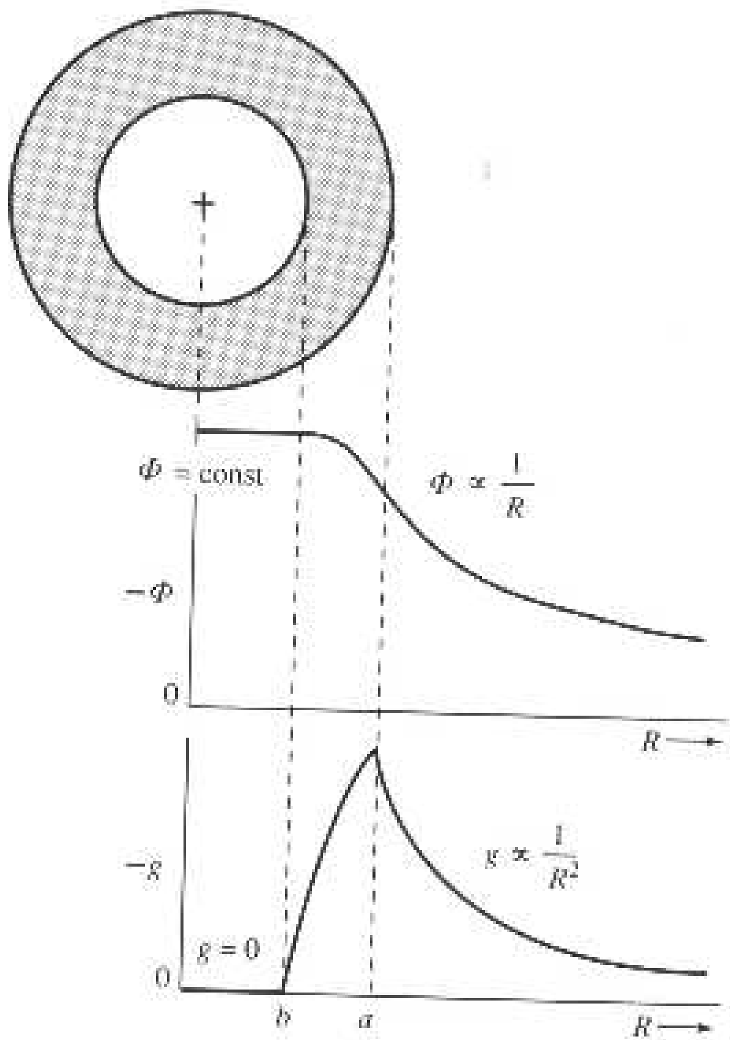
$\Downarrow$

$$g = -\frac{d\Phi}{dR}$$

$$\left.\begin{array}{rcc} g(R<b) & = & 0 \\ g(b<R<a) & = & \frac{4\pi\rho G}{3}\left(\frac{b^3}{R^2} - R\right) \\ g(R>a) & = & -\frac{GM}{R^2} \end{array}\right\}$$

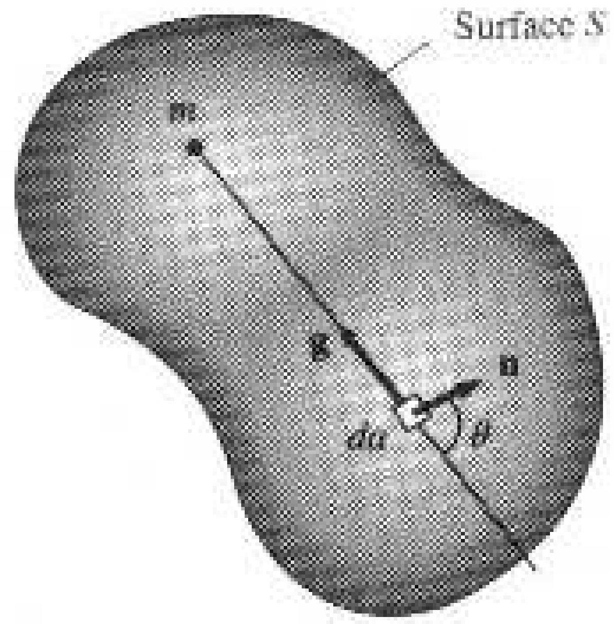
Gravitational Potential (cont'd)

SUMMARY II



Poisson's Equation

The gravitational flux Φ_m emanating from a mass m



through the arbitrary surface S is

$$\Phi_m = \int_S \vec{n} \cdot \vec{g} \, da$$

$\vec{n}$ is normal to the surface at the differential area da

Poisson's Equation (cont'd)

Substituting $\vec{g}$ for the gravitational vector field for a body of mass m

$$\vec{n} \cdot \vec{g} = -Gm \frac{\cos\theta}{r^2} \qquad \theta \text{ angle between } \vec{n} \text{ and } \vec{g}$$

$$\Downarrow$$

$$\Phi_m = -Gm \int_S \frac{\cos\theta}{r^2} \, da$$

The integral is over the solid angle of the arbitrary surface

$$\Phi_m = \int_S \vec{n} \cdot \vec{g} \, da = -4\pi Gm$$

Generalizing this result for m_i masses

$$\int_S \vec{n} \cdot \vec{g} \, da = -4\pi G \sum_i m_i$$

and for a continuos mass distribution within surface S

$$\int_S \vec{n} \cdot \vec{g} \, da = -4\pi G \int_V \rho \, dv$$

V is the volume enclose by S and ρ the mass density

Poisson's Equation (cont'd)

Using Gauss divergence theorem

$$\int_S \vec{n} \,.\, \vec{g} \, da = \int_V \vec{\nabla} \,.\, \vec{g} \, dv$$

$$\Downarrow$$

$$\int_V -4\pi G\rho \, dv = \int_V \vec{\nabla} \,.\, \vec{g} \, dv$$

or equivalently

$$\vec{\nabla} \,.\, \vec{g} = -4\pi G\rho$$

Substituting $\vec{g} = -\vec{\nabla}\Phi$ ☚ we obtain Poisson's Equation

$$\vec{\nabla}^2\Phi = -4\pi G\rho$$

If $\rho = 0$ ☚ we obtain Laplace Equation

$$\vec{\nabla}^2\Phi = 0$$

Ocean Tides

The ocean tides have long been of interest to humans

The Chinese explained the tides as the breathing of the Earth

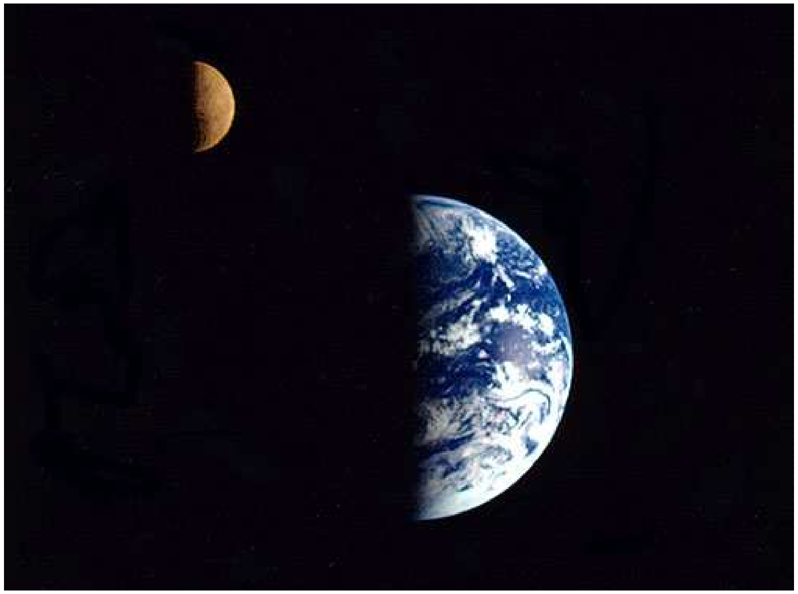
☞ around the sun once a year

Galileo tried unsuccessfully to explain the tides ☚ effect of Earth's motion

☞ on its own axis once a day

(could not account for the timing of the approximately two high tides each day)

Mariners have known for at least 4000 yr that tides are related to Moon's phases



Exact relationship ☚ **hidden behind many complicated factors**

Newton finally gave an adequate explanation

Ocean Tides (cont'd)

☞ to the moon

- **Ocean tides are caused by the gravitational attraction of the ocean**

☞ to the sun

Calculation is complicated

$\Downarrow$

surface of the Earth is not an inertial system !

☞ **Earth's rotation**

- **Timing of tidal events is related to**

☞ **revolution of the moon around the Earth**

If the moon was stationary in space ☚ tidal cycle would be 24 hours long

HOWEVER

the moon is in motion revolving around the Earth

1 revolution takes about 27 days

$\Downarrow$

adding about 50 minutes to the tidal cycle

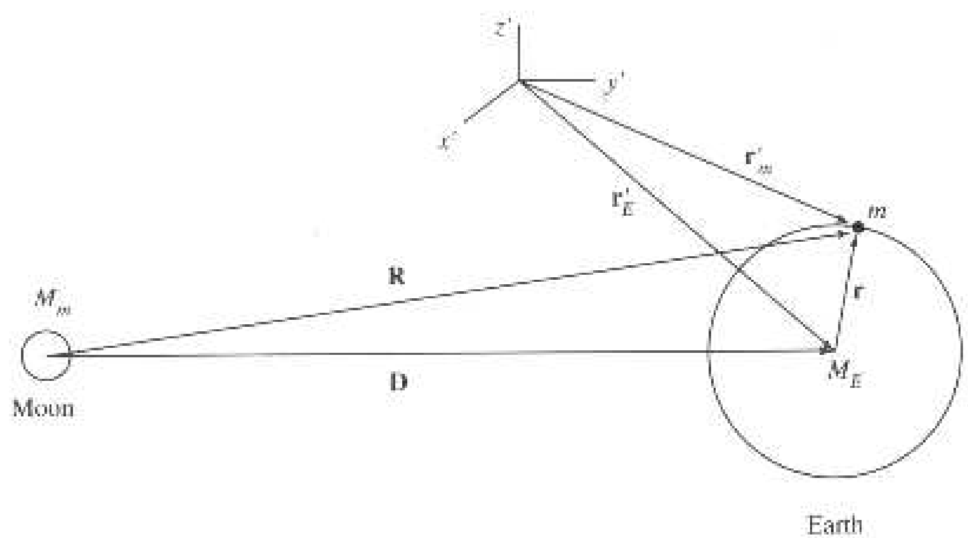
Ocean Tides (cont'd)

1^{st} ☞ consider only effect of the moon

⇓

assume simple model ☞ Earth's surface is completely full of water

set up an inercial frame of reference x', y', z'



$M_{\rm m} \equiv$ mass of the moon

$r \equiv$ radius of a circular Earth

$D \equiv$ distance from the center of the Earth to the center of the moon

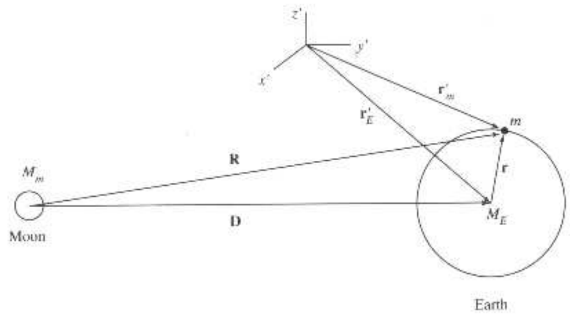
Ocean Tides (cont'd)

Consider the effect of both moon and Earth gravitational attraction

☞ on a small mass m placed on the surface of the Earth

The position vector of the mass m
- from the center of the Earth is $\vec{r}$
- from the moon is $\vec{R}$
- from our inertial system $\vec{r}'_m$

The position vector from the inertial system to the center of the Earth is $\vec{r}'_{\rm E}$



As measured from inertial system ☚ the force on m due to Earth and moon is

$$m \;\; \ddot{\vec{r}}'_m = -\frac{G\, m\, M_{\rm E}}{r^2}\; \vec{e}_r - \frac{G\, m\, M_{\rm m}}{R^2}\; \vec{e}_R$$

Similarly ☞ the force on the center of mass of the Earth caused by the moon is

$$M_E \;\; \ddot{\vec{r}}'_{\rm E} = -\frac{G\, M_{\rm E}\, M_{\rm m}}{D^2}\; \vec{e}_D$$

Ocean Tides (cont'd)

We want to find ☞
acceleration as measured in the noninertial frame place at center of Earth

⇓

$$
\begin{aligned}
\ddot{\vec{r}} &= \ddot{\vec{r}}'_m - \ddot{\vec{r}}'_{\rm E} \\
&= \frac{m \quad \ddot{\vec{r}}'_m}{m} - \frac{M_{\rm E} \quad \ddot{\vec{r}}'_{\rm E}}{M_{\rm E}} \\
&= -\frac{G\,M_{\rm E}}{r^2}\,\vec{e}_r - \frac{G\,M_{\rm m}}{R^2}\,\vec{e}_R + \frac{G\;M_{\rm m}}{D^2}\,\vec{e}_D \\
&= -\frac{G\,M_{\rm E}}{r^2}\,\vec{e}_r - G\,M_m\left(\frac{\vec{e}_R}{R^2} - \frac{\vec{e}_D}{D^2}\right)
\end{aligned}
$$

- **The first part is due to the Earth**
- **The second part is the acceleration from the tidal force**

⇓ ☞ the moon's gravitational pull at the center of the Earth

due to the difference between

☞ the moon's gravitational pull on the Earth surface

Ocean Tides (cont'd)

We next find the effect of the tidal force at various points on Earth

Polar view of Earth with the axis along z-axis

The tidal force on the mass m on the Earth surface is

$$F_{\rm T} = -G\, m\, M_{\rm m} \left(\frac{\vec{e}_R}{R^2} - \frac{\vec{e}_D}{D^2} \right)$$

For the farthest point on Earth from the moon ☛ a

both unit vectors $\vec{e}_R$ and $\vec{e}_D$ are pointing in the same direction

(away from the moon in the x-axis)

Since $R > D$ the second term predominates ☞ the force is along the $+\,x$-axis

For the closest point on Earth from the moon ☛ b

$R < D$ and $r/D \ll 1$ $\Downarrow$

$\vec{F}_{\rm T}$ has approximately the same magnitude as at point a ☛ but along the $-x$-axis

Ocean Tides (cont'd)

The magnitude of the tidal force along the x-axis is

$$F_{\mathrm{T}x} = -G\,m\,M_{\mathrm{m}}\left(\frac{1}{R^2}-\frac{1}{D^2}\right)$$
$$= -G\,m\,M_{\mathrm{m}}\left[\frac{1}{(D+r)^2}-\frac{1}{D^2}\right]$$
$$= -G\,m\,M_{\mathrm{m}}\frac{1}{D^2}\left[\frac{1}{(1+r/D)^2}-1\right]$$

We expand first term using $(1+x)^{-2}$ expansion

$$F_{\mathrm{T}x} = -\frac{G\,m\,M_{\mathrm{m}}}{D^2}\left[1-2\frac{r}{D}+3\left(\frac{r}{D}\right)^2-\ldots-1\right]$$
$$= \frac{2GmM_{\mathrm{m}}r}{D^3}$$

we kept only the largest non-zero term in the expansion because $r/D = 0.02$

Ocean Tides (cont'd)

For point c ☞ though $\vec{e}_R$ is not quite exactly along $\vec{e}_D$
the x-axis components approximately cancel because $R \simeq D$
and the x-components of e_R and e_D are similar
There is a small component of e_R along y-axis ☞ can be approximated by $(r/D)\hat{\jmath}$

$\downarrow$

Tidal force at point c is

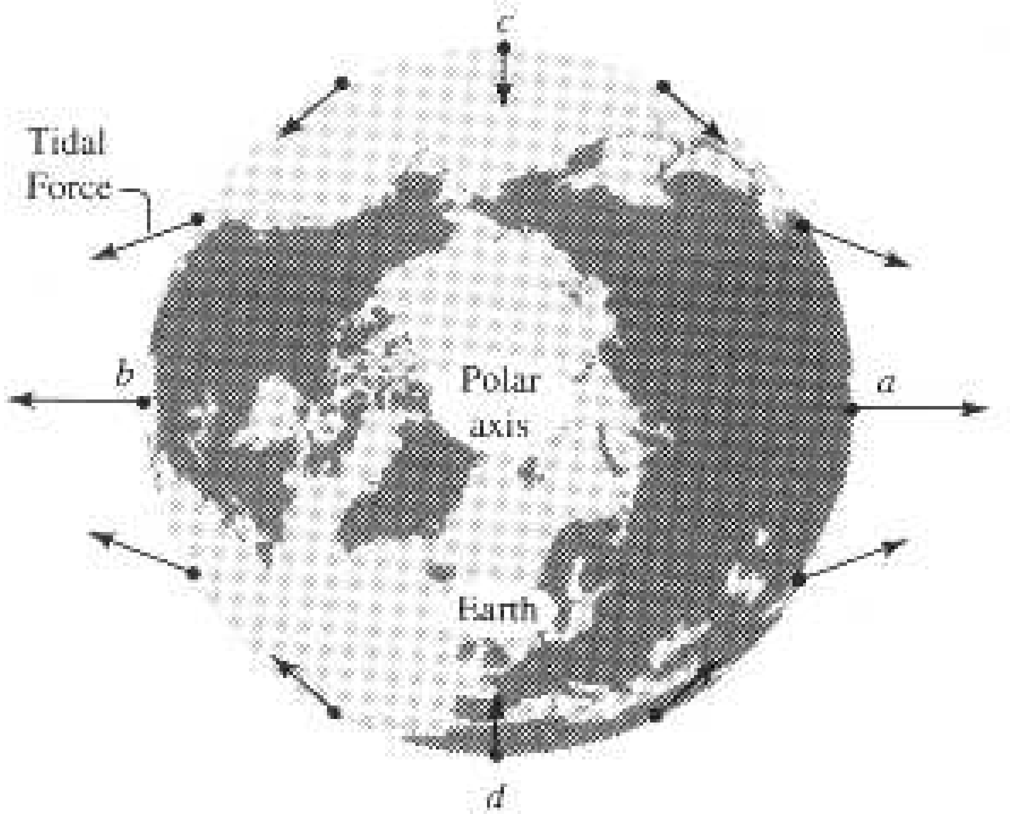
$$F_{\mathrm{T}y} = -GmM_{\mathrm{m}} \left(\frac{1}{D^2} \frac{r}{D} \right) = -\frac{GmM_{\mathrm{m}} r}{D^3}$$

This force is along the $-y$-axis towards the center of the Earth at point c

For point d ☞ force of same magnitude
☛ but along the $+y$-axis towards the center of the Earth

Ocean Tides (cont'd)

SUMMARY



Ocean Tides (cont'd)

x- and y- components of $\vec{F}_{\rm T}$ can be found by substituting x and y for r in $F_{{\rm T}x}$

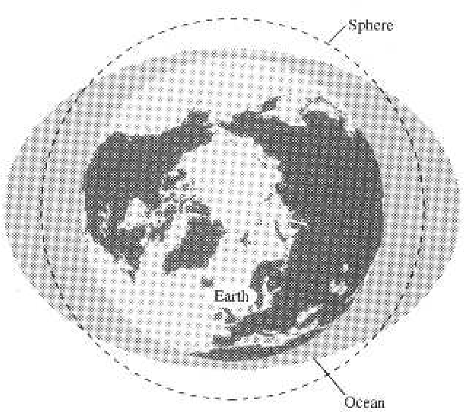
$$F_{{\rm T}x} = \frac{2GmM_{\rm m}x}{D^3}$$

$$F_{{\rm T}y} = -\frac{GmM_{\rm m}y}{D^3}$$

For arbitrary point e ☞ $x = r\cos\theta$ and $y = r\sin\theta$

$$F_{{\rm T}x} = \frac{2GmM_{\rm m}r\cos\theta}{D^3}$$

$$F_{{\rm T}y} = -\frac{GmM_{\rm m}r\sin\theta}{D^3}$$



Ocean Tides (cont'd)

The plane of the moon orbit is not perpendicular to the Earth rotation axis

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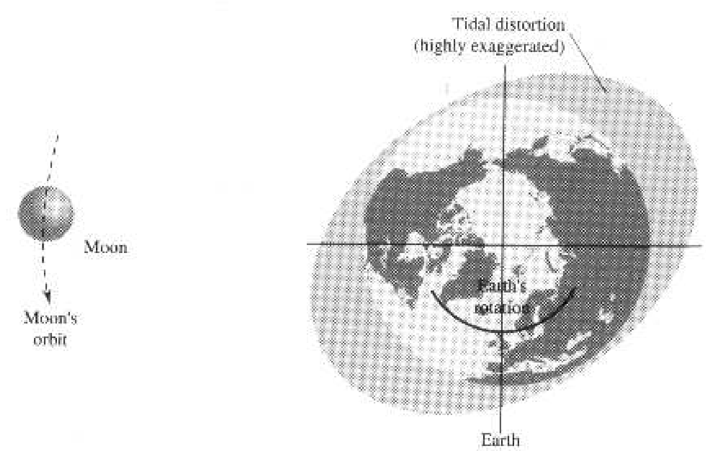
high tides are not along the Earth-moon axis



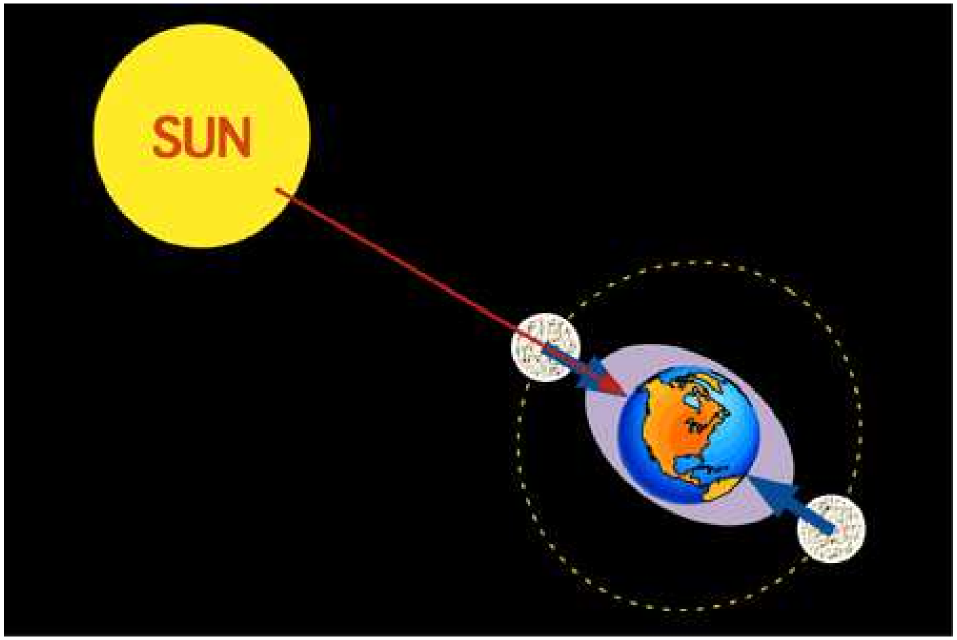
The sun gravitational attraction is about 175 times stronger
Despite the stronger attraction due to the sun ☞
gravitational force gradient over the surface of the Earth is much smaller
(because of much larger distance to the sun)
Similar analysis shows ☛ tidal force due to the sun is 46% that of moon
sizeable effect!

Ocean Tides (cont'd)

Second factor controlling tides on Earth's surface is the sun's gravity

The height of the average solar tide is about 0.46 the average lunar tide

At certain times during the moon's revolution around the Earth ☞
the direction of its gravitational attraction is aligned with the sun's



during these times the two tide producing bodies act together

⇓

creating the highest and lowest tides of the year

Spring tides occur every 14-15 days during full and new moons

Ocean Tides (cont'd)

When gravitational pull of moon and sun are at right angles to each other

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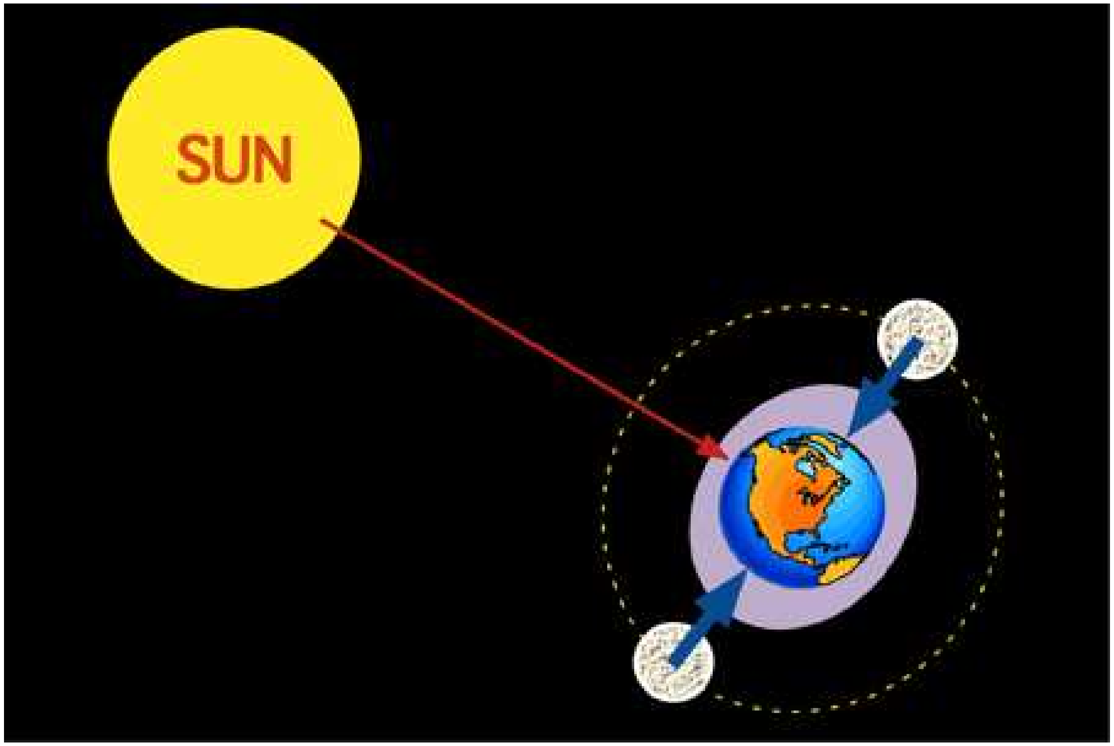
daily tidal variations on the Earth are at their least



Neap tides occur during the first and last quarter of the moon

Homework

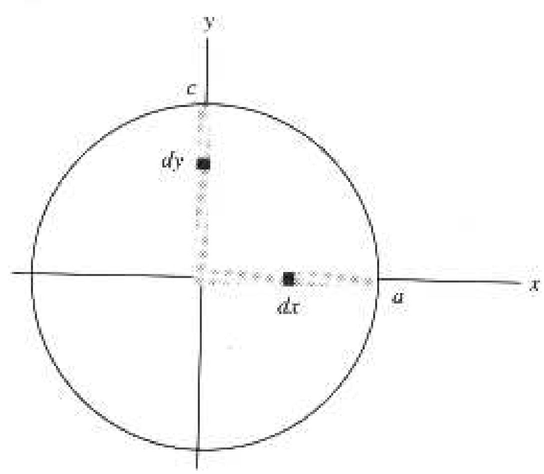
Using the simple model of ocean surrounding the Earth
calculate the maximum height change in the ocean tides

Newton's hint

Imagine 2 wells be dug

☞ one along the direction of high tide (x-axis)

☞ one along the direction of low tide (y-axis)



If tidal height change by h ⇓
difference in potential energy of mass m due to height difference is mgh
ΔW done to move mass m from point c to center of the Earth and then to point a
☞ must equal the potential energy change mgh

Homework (cont'd)

work done by gravity

$$W = \int_r^0 F_{\mathrm{T}y}\, dy + \int_0^r F_{\mathrm{T}x}\, dx$$

$$\begin{aligned} W &= \frac{GmM_{\mathrm{m}}}{D^3}\left[\int_r^0 (-y)dy + \int_0^r 2x dx\right] \\ &= \frac{3GmM_{\mathrm{m}}r^2}{2D^3} \end{aligned}$$

$$mgh = \frac{3GmM_{\mathrm{m}}r^2}{2D^3} \Rightarrow h = \frac{3GM_{\mathrm{m}}r^2}{2gD^3}$$

$$h = \frac{3(6.67\times 10^{-11}\ \mathrm{m}^3/\mathrm{kg\,s}^2)\ (7.350\times 10^{22}\ \mathrm{kg})\ (6.37\times 10^6\ \mathrm{m})^2}{2\ (9.8\ \mathrm{m/s}^2)\ (3.84\times 10^8\ \mathrm{m})^3} = 0.54\ \mathrm{m}$$

RU ok with this answer?

Homework (cont'd)

work done by gravity

$$W = \int_r^0 F_{\mathrm{T}y}\, dy + \int_0^r F_{\mathrm{T}x}\, dx$$

$$\begin{aligned} W &= \frac{GmM_{\mathrm{m}}}{D^3}\left[\int_r^0 (-y)dy + \int_0^r 2x dx\right] \\ &= \frac{3GmM_{\mathrm{m}}r^2}{2D^3} \end{aligned}$$

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An observer who has spent much time near the ocean has noticed that typical oceanshore tides are much greater

$\Downarrow$

Earth is not covered completely with water and continents play a significant role
local effects can be dramatic ☞ leading to tidal changes of several meters!