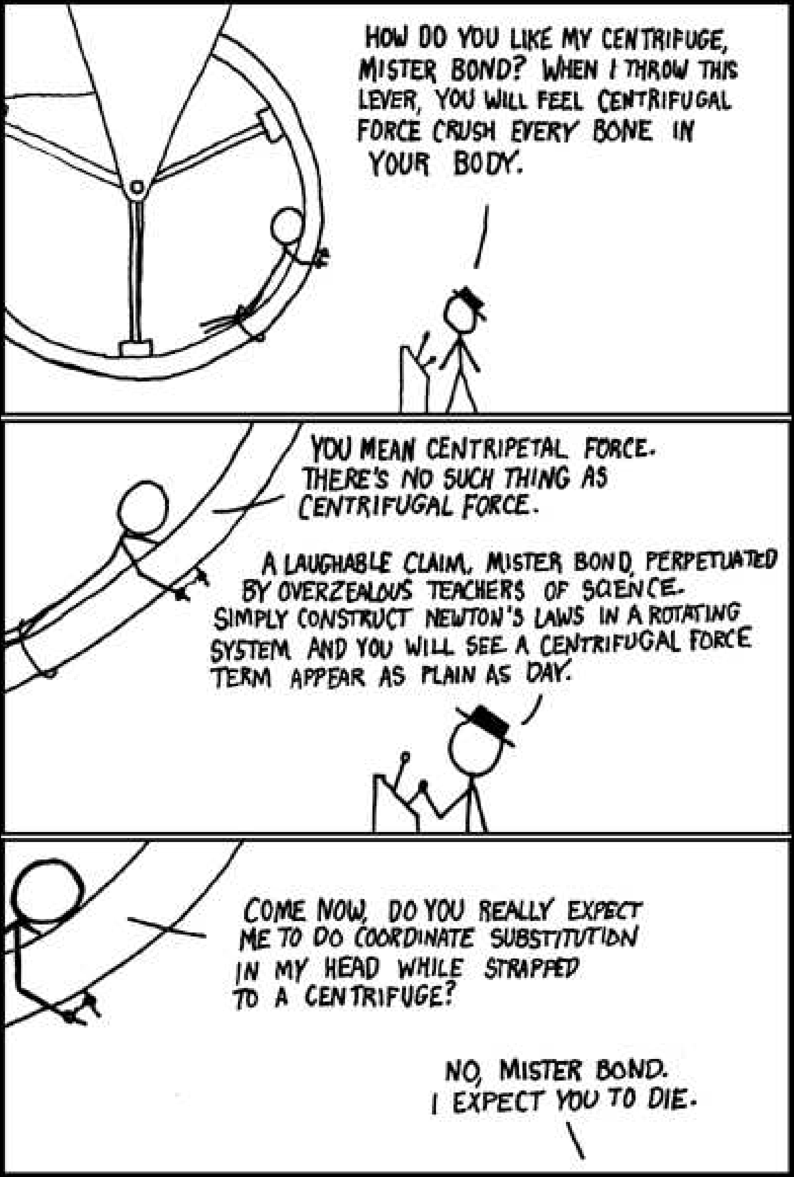
Classical Mechanics

Non-Inertial Reference Frames

- ☞ Rotating Reference Frames
- ☞ Centrifugal Acceleration
- ☞ The Coriolis Force

Fictitious Forces with Licence to Kill



Rotating Reference Frames

- $\vec{r}$ ☞ position vector of an object in some non-rotating inertial reference frame
- observe the motion of such object in a non-inertial reference frame which rotates with constant angular velocity $\vec{\Omega}$ about an axis passing through the origin of the inertial frame

If the object appears stationary in the rotating reference frame

$$\Downarrow$$

in the non-rotating frame ☞ the object's position vector $\vec{r}$ will appear to precess about the origin with angular velocity $\vec{\Omega}$

$$\Downarrow$$

in the non-rotating reference frame

$$\frac{d\vec{r}}{dt} = \vec{\Omega} \times \vec{r}$$

If now the object appears to move in the rotating reference frame with instantaneous velocity $\vec{v}\,'$

It is fairly obvious that the appropriate generalization of the above equation is

$$\frac{d\vec{r}}{dt} = \vec{v}\,' + \vec{\Omega} \times \vec{r}$$

Rotating Reference Frames (cont'd)

- Let
  - d/dt denote apparent time derivatives in the non-rotating frame of reference
  - d/dt' denote apparent time derivatives in the rotating frame of reference

Since an object which is stationary in the rotating reference frame appears to move in the non-rotating frame ☞ $d/dt \neq d/dt'$

Writing the apparent velocity $\vec{v}\,'$ in the rotating reference frame as $d\vec{r}/dt'$ the equation of motion takes the form

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{dt'} + \vec{\Omega} \times \vec{r}$$

or ☞ since $\vec{r}$ is a general position vector

$$\Downarrow$$

$$\frac{d}{dt} = \frac{d}{dt'} + \vec{\Omega}\times$$

This equation expresses the relationship between apparent time derivatives in the non-rotating and rotating reference frames

Rotating Reference Frames (cont'd)

Operating on the general position vector $\vec{r}$ with the time derivative we get

$$\vec{v} = \vec{v}\,' + \vec{\Omega} \times \vec{r}$$

This equation relates the apparent velocity $\vec{v} = d\vec{r}/dt$
of an object with position vector $\vec{r}$ in the non-rotating reference frame
to its apparent velocity $\vec{v}\,' = d\vec{r}/dt'$ in the rotating reference frame

Operating twice on the position vector $\vec{r}$ with the time derivative we obtain

$$\vec{a} = \left(\frac{d}{dt'} + \vec{\Omega}\times\right)\left(\vec{v}\,' + \vec{\Omega} \times \vec{r}\right)$$

or equivalently

$$\vec{a} = \vec{a}' + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + 2\,\vec{\Omega} \times \vec{v}\,'$$

This equation relates the apparent acceleration $\vec{a} = d^2\vec{r}/dt^2$
of an object with position vector $\vec{r}$ in the non-rotating reference frame
to its apparent acceleration $\vec{a}' = d^2\vec{r}/dt'^2$ in the rotating reference frame

Rotating Reference Frames (cont'd)

Applying Newton's Second Law of motion in the inertial reference frame
(i.e., non-rotating)

$$\Downarrow$$

$$m\,\vec{a} = \vec{f}$$

- m ☞ mass of the object
- $\vec{f}$ ☞ (non-fictitious) force acting on it

Note that these quantities are the same in both reference frames

The apparent equation of motion in the rotating reference frame takes the form

$$m\,\vec{a}' = \vec{f} - m\,\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) - 2\,m\,\vec{\Omega} \times \vec{v}\,'$$

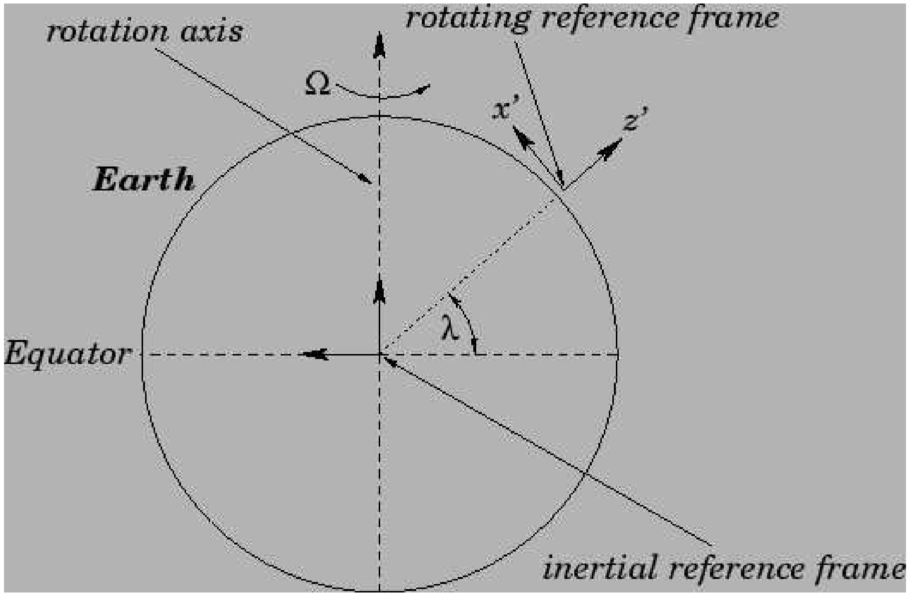
The last two terms in the above equation are so-called "fictitious forces"
(always needed to account for motion observed in non-inertial reference frames)
Let us now investigate the two fictitious forces in detail

Centrifugal Acceleration

Take a non-rotating inertial frame whose origin lies at the center of the Earth
Consider a rotating frame whose origin is fixed with respect to some point of latitude λ on the Earth's surface

The latter reference frame thus rotates with respect to the former (about an axis passing through the Earth's center) with an angular velocity vector $\vec{\Omega}$ which points from the center of the Earth towards its North Pole and is of magnitude

$$\Omega = \frac{2\pi}{24 \text{ hrs}} = 7.27 \times 10^{-5} \, \text{rad./s}$$



Centrifugal Acceleration (cont'd)

Consider an object which appears stationary in our rotating reference frame
(i.e., an object which is stationary with respect to the Earth's surface)
The object's apparent equation of motion in the rotating frame takes the form

$$m\,\vec{a}' = \vec{f} - m\,\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

Let the non-fictitious force acting on our object be the force of gravity $\vec{f} = m\,\vec{g}$
The local gravitational acceleration $\vec{g}$ points towards the center of the Earth

$$\Downarrow$$

the apparent gravitational acceleration in the rotating frame is written

$$\vec{g}' = \vec{g} - \vec{\Omega}\times(\vec{\Omega}\times\vec{R})$$

- The apparent gravitational acceleration of a stationary object has 2-components
  - the true gravitational acceleration $\vec{g}$ of magnitude $g \sim 9.8\,\mathrm{m/s^2}$ which always points directly towards the center of the Earth
  - the so-called centrifugal acceleration ☞ $-\vec{\Omega}\times(\vec{\Omega}\times\vec{R})$ which is normal to the Earth's axis of rotation pointing away from this axis

The magnitude of the centrifugal acceleration is $\Omega^2\,\rho = \Omega^2\,R\cos\lambda$

- ρ ☞ perpendicular distance to the Earth's rotation axis
- $R = 6.37\times 10^6\,\mathrm{m}$ is the Earth's radius

The Coriolis Force

Next we investigate the second "fictitious force" called the Coriolis force

which only affects objects which are moving in the rotating reference frame

Consider a particle of mass m free-falling under gravity in a rotating frame

- Define Cartesian axes in the rotating frame such that
  - the z'-axis points vertically upward
  - the x'- and y'-axes are horizontal
  - x'-axis pointing directly northwards and y'-axis pointing directly westward

The Cartesian equations of motion of the particle in the rotating reference frame take the form

$$\ddot{x}' = 2\,\Omega\,\sin\lambda\,\dot{y}$$

$$\ddot{y}' = -2\,\Omega\,\sin\lambda\,\dot{x}' + 2\,\Omega\,\cos\lambda\,\dot{z}'$$

$$\ddot{z}' = -g - 2\,\Omega\,\cos\lambda\,\dot{y}'$$

- $(\dot{}) \equiv d/dt$
- g ☞ local acceleration due to gravity

we have neglected the centrifugal acceleration for the sake of simplicity

this is reasonable ☞ the only effect of the centrifugal acceleration is to slightly modify the magnitude and direction of the local gravitational acceleration

The Coriolis Force (cont'd)

Consider a particle which is dropped (at $t = 0$) from rest
a height h above the Earth's surface

To lowest order the particle's vertical motion satisfies (i.e., neglecting Ω)

$$z' = h - \frac{g\,t^2}{2}$$

Substituting this expression into the equations of motion
neglecting terms involving Ω^2

$$x' \simeq 0 \qquad\qquad y' \simeq -g\,\Omega\,\cos\lambda\,\frac{t^3}{3}$$

The particle is deflected eastward ☞ in the negative y'-direction
the particle hits the ground when $t \simeq \sqrt{2\,h/g}$
the net eastward deflection of the particle as strikes the ground is

$$d_{\rm east} = \frac{\Omega}{3}\,\cos\lambda\,\left(\frac{8\,h^3}{g}\right)^{1/2}$$

This deflection is in the same direction as the Earth's rotation ☞ West to East
It is greatest at the Equator and zero at the Poles
A particle dropped from a height of 100m at the Equator
is deflected by about $2.2\,{\rm cm}$

The Coriolis Force (cont'd)

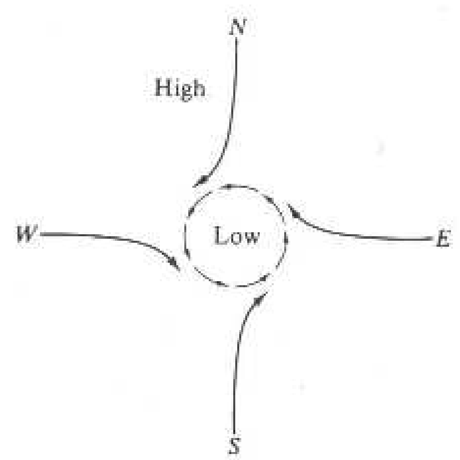
The Coriolis force has a significant effect on terrestrial weather patterns

Near equatorial regions
the intense heating of the Earth's surface due to the Sun
results in hot air rising

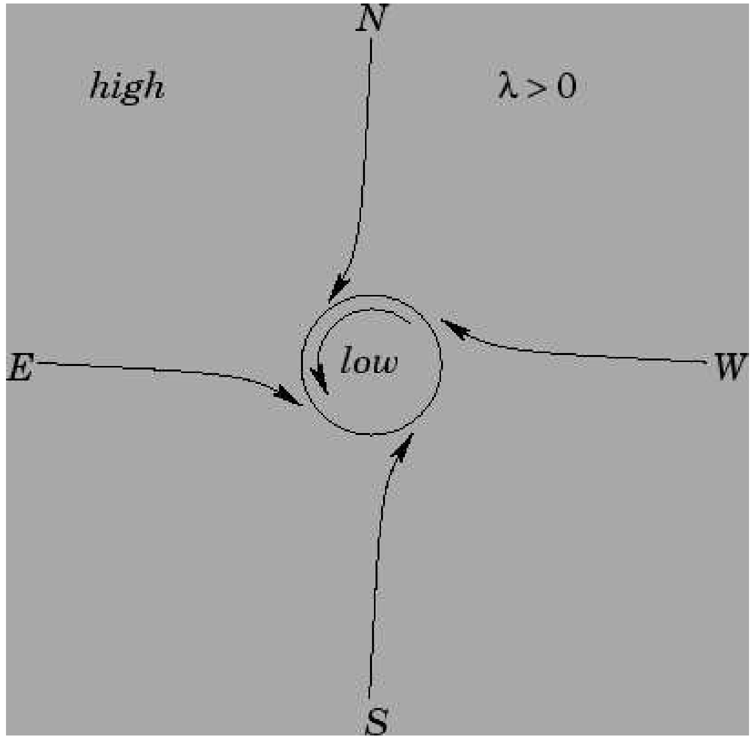
In the Northern Hemisphere
this causes cooler air to move in a southerly direction towards the Equator

The Coriolis force deflects this moving air in a clockwise sense
(looking from above)
resulting in the trade winds which blow towards the southwest

In the Southern Hemisphere the cooler air moves northwards
and is deflected by the Coriolis force in an anti-clockwise sense
resulting in trade winds which blow towards the northwest



The Coriolis Force (cont'd)



As air flows from high to low pressure regions
the Coriolis force deflects the air in a clockwise/anti-clockwise manner
in the Northern/Southern Hemisphere producing cyclonic rotation

It follows that cyclonic rotation is

- anti-clockwise in the Northern Hemisphere
- clockwise in the Southern Hemisphere

This is the direction of rotation of tropical storms (hurricanes) in each hemisphere