Classical Mechanics

Fundamental aspects of Newton's theory of motion

- ☞ Newton's Laws ✓
- ☞ Inclined Plane ✓
- ☞ Projectiles ✓
- ☞ Conservation Theorems ✓
- ☞ Rocket Motion ✓
- ☞ Motion in a General 1-dimensional Potential ✓
- ☞ Oscillations

Hooke's Law

Consider the motion of an object (of mass m) which is slightly perturbed from a stable equilibrium point (at $x = 0$) of the conservative force-field $f(x)$

For $x = 0$ to be a stable equilibrium point ☛ $f(0) = 0$ and $df(0)/dx < 0$

The object obeys Newton's Second Law of motion ☛ $m\,\ddot{x} = f(x)$

If it always stays fairly close to its equilibrium position (to a good approximation)

$$\Downarrow$$

$f(x)$ can be represented by a truncated Taylor series about this position

$$f(x) \simeq f(0) + \frac{df}{dx}(0)\,x + \mathcal{O}(x^2)$$

$-df(0)/dx = k$ ☛ restoring force is always directed to the equilibrium position

$$\Downarrow$$

derivative is negative $\Rightarrow$ k is a positive constant

The equivalent of this force law was originally announced in 1676
by Robert Hooke in the form of a Latin cryptogram

CEIIINOSSSTTUV

Hooke later provided a translation

ut tensio sic vis ☛ the stretch is proportional to the force

The Simple Harmonic Oscillator

Equation of motion for the simple harmonic oscillator

$\Downarrow$

Substitution of Hooke's law force into the Newtonian equation
(with $df(0)/dx = -m\,\omega_0^2$)

$$\frac{d^2x}{dt^2} + \omega_0^2\, x \simeq 0$$

Solution governs the motion of all 1-dimensional conservative systems which are slightly perturbed from some stable equilibrium point

$$x(t) = A\,\sin(\omega_0\, t - \delta)$$

$$x(t) = A\,\cos(\omega_0\, t - \phi)$$

Pattern of motion is periodic in time with repetition period $\tau_0 = 2\pi/\omega_0$
oscillating between $x = \pm A$
Phase angle simply shifts pattern of motion backward and forward in time
$(\delta - \phi = \pi/2)$

The Simple Harmonic Oscillator (cont'd)

Relation between the kinetic energy and the amplitude of motion

$$\begin{aligned} T &= \frac{1}{2} m\dot{x}^2 \\ &= \frac{1}{2} m\omega_0^2 A^2 \cos^2(\omega_0 t - \delta) \\ &= \frac{1}{2} kA^2 \cos^2(\omega_0 t - \delta) \end{aligned}$$

Potential energy calculated from work done to displace particle distance x

$$dW = -F\,dx = kxdx$$

$$\Downarrow$$

$$U = \frac{1}{2} kx^2$$

The Simple Harmonic Oscillator (cont'd)

The potential energy at position x is

$$U(x) \simeq \frac{1}{2}\, m\, \omega_0^2\, x^2$$

The total (mechanical) energy is

$$\begin{aligned} E &= T + U \\ &= \frac{1}{2}\, m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\, m\, \omega_0^2\, x^2 \\ &= \frac{1}{2}\, m\, \omega_0^2\, A^2\, \cos^2(\omega_0\, t - \phi) + \frac{1}{2}\, m\, \omega_0^2\, A^2\, \sin^2(\omega_0\, t - \phi) \\ &= \frac{1}{2}\, m\, \omega_0^2\, A^2 \end{aligned}$$

angular frequency of the motion (ω_0) is related to the frequency (ν_0) by

$$\omega_0 = 2\pi\nu_0 = \sqrt{k/m}$$

☞ independent of the amplitude

- The period of the simple harmonic oscillator is

☞ independent of the total energy

- A system exhibiting this property is said to be **isochronous**

Phase Diagrams

State of motion of 1-dimensional oscillator is completely specified by 2 quantities $x(t)$ and $\dot{x}(t)$
(two quantities needed because differential equation of motion is second order)

$x(t)$ and $\dot{x}(t)$ define coordinates of points in a 2-dimensional space

$\Downarrow$

The Phase Space

In two dimensions the phase space is a phase plane

for general oscillator with n degrees of freedom ☞ 2n-dimensional phase space

- As the time varies the point $P(x, \dot{x})$ describing the state of the oscillating particle will move along a certain phase path in the phase plane
- For different initial conditions the motion will be described by different paths
- The totality of all phase paths constitutes the phase diagram of the oscillator

Phase Diagrams (cont'd)

simple harmonic oscillator

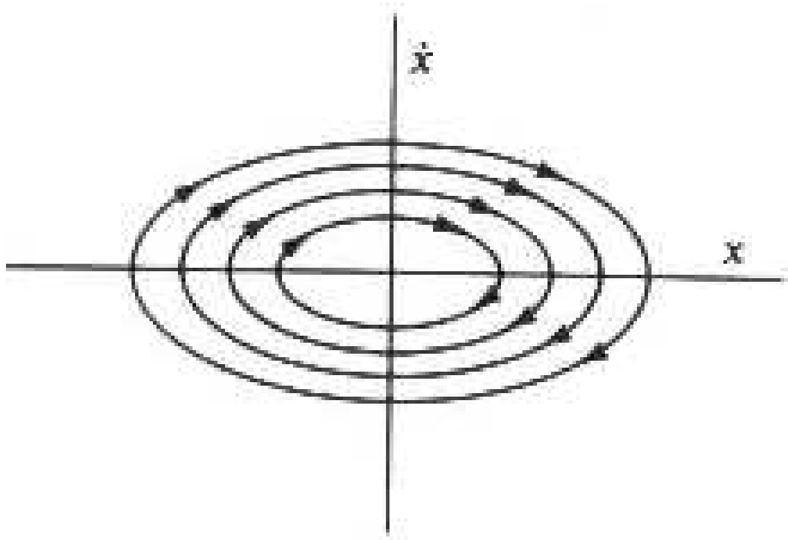
$$x(t) = A\sin(\omega_0 t - \delta)$$

$$\dot{x}(t) = A\omega_0 \cos(\omega_o t - \delta)$$

Eliminating t

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{A^2\omega_0^2} = 1$$

This equation represents a family of elipses



Phase Diagrams (cont'd)

Total Energy $E = kA^2/2$ Angular Frequency $\omega_0^2 = k/m$

$$\Downarrow$$

$$\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1$$

Each phase path corresponds to a definite total energy of the oscillator

No two phase paths of the oscillator can cross!!!

If they could cross ☞ this would imply that for a given set of initial conditions the motion could proceed along different paths

$$\Downarrow$$

This is impossible since the solution of the differential equation is unique

Harmonic Oscillations in 2-dimensions

Consider the motion of a particle with two degrees of freedom

☞ proportional to the distance of the particle from a force center (located at the origin)

- Restoring force

  ☞ directed toward the origin

$$\vec{F} = -k\vec{r}$$

in polar coordinates components

$$\left.\begin{array}{l} F_x = -kr\cos\theta = -kx \\ F_y = -kr\sin\theta = -ky \end{array}\right\}$$

Equation of Motion

$$\left.\begin{array}{l} \ddot{x} + \omega_0 x = 0 \\ \ddot{y} + \omega_0 y = 0 \end{array}\right\}$$

recall ☞ $\omega_0^2 = k/m$

Solution

$$\left.\begin{array}{l} x(t) = A\cos(\omega_0 t + \alpha) \\ y(t) = B\cos(\omega_0 t + \beta) \end{array}\right\}$$

Harmonic Oscillations in 2-dimensions (cont'd)

Equation for the path of the particle

$\Downarrow$

eliminate t between the 2 equations

$$\begin{aligned} y(t) &= B\cos[\omega_0 t - \alpha + (\alpha - \beta)] \\ &= B\cos(\omega_0 t - \alpha)\,\cos(\alpha-\beta) - B\sin(\omega_0 t - \alpha)\,\sin(\alpha - \beta) \end{aligned}$$

Define $\delta \equiv \alpha - \beta$ and recall ☞ $\cos(\omega_0 t - \alpha) = x/A$

$$y = \frac{B}{A}\, x\cos\delta - B\sqrt{1-(x/A)^2}\,\sin\delta$$

$$Ay - Bx\,\cos\delta = -B\sqrt{A^2 - x^2}\,\sin\delta$$

on squaring

$$A^2y^2 - 2A\,By\,x\,\cos\delta + B^2x^2\,\cos^2\delta = A^2B^2\sin^2\delta - B^2x^2\,\sin^2\delta$$

Harmonic Oscillations in 2-dimensions (cont'd)

$$B^2x^2 - 2ABxy\cos\delta + A^2y^2 = A^2B^2\sin^2\delta$$

$$\Downarrow$$

$$\text{If } \delta = \pm\pi/2 \Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

If further require $A = B \Rightarrow$ circular motion

$$x^2 + y^2 = A^2$$

If the phase $\delta = n\pi$ with $n \in \mathbb{Z}$

$$B^2x^2 - 2ABxy + A^2y^2 = 0 \Rightarrow (Bx - Ay)^2 = 0$$

$$\Downarrow$$

linear solution

$$\Downarrow$$

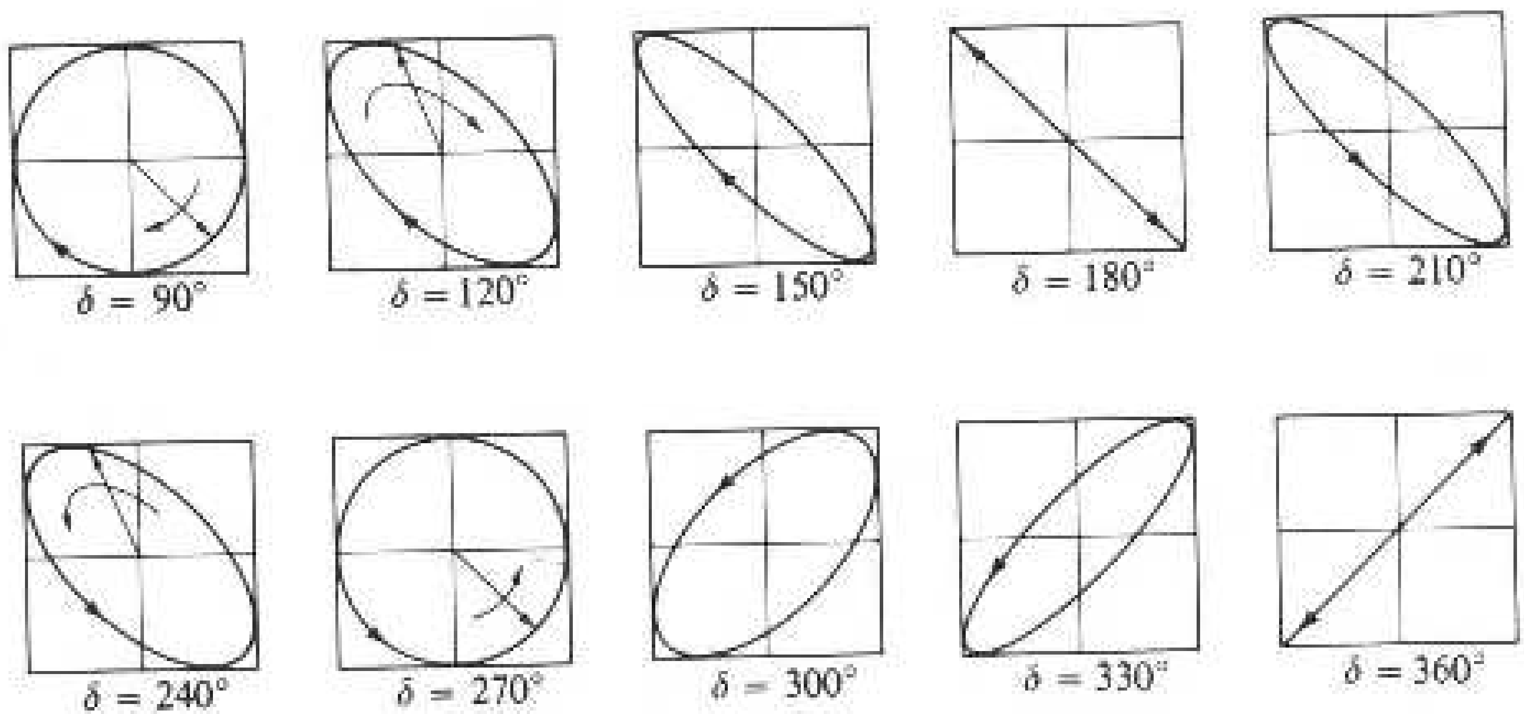
$$y = \frac{B}{A}x \Leftrightarrow \delta = 0 \qquad\qquad y = -\frac{B}{A}x \Leftrightarrow \delta = \pm\pi$$

Harmonic Oscillations in 2-dimensions (cont'd)

SUMMARY

If $A = B$

$$\Downarrow$$



Damped Oscillatory Motion

1-D conservative systems slightly perturbed from a stable equilibrium point

(and then left alone)

oscillate about this point with a fixed frequency and a constant amplitude

$\Downarrow$

In other words ☞ the oscillations never die away

This is not very realistic ☞ (in practice)

If we slightly perturb a dynamical system from a stable equilibrium point

$\Downarrow$

it will indeed oscillate about this point ☞ but these oscillations will eventually die away due to frictional effects that are present in all real dynamical systems

In order to model frictional effects

need to include ☞ frictional drag force in our perturbed equation of motion

☛ is always directed in the opposite direction to the instantaneous velocity

The most common model for a frictional drag force is one which

☛ is directly proportional to the magnitude of this velocity

Damped Oscillatory Motion (cont'd)

drag force can be written

$$f_{\rm drag} = -2\, m\, \beta\, \frac{dx}{dt}$$

β ☞ positive constant.

Including such a force in our perturbed equation of motion

$$\Downarrow$$

$$\frac{d^2x}{dt^2} + 2\,\beta\,\frac{dx}{dt} + \omega_0^2\, x = 0$$

$$\Downarrow$$

this is a linear second-order ordinary differential equation

(that we suspect possesses oscillatory solutions)

☛ β parameterizes the strength of frictional damping in our dynamical system

Damped Oscillatory Motion (cont'd)

There is a standard trick for solving such an equation

$$\Downarrow$$

search for complex oscillatory solutions of the form

$$x = A\,\mathrm{e}^{-\mathrm{i}\,\omega\,t}$$

ω and A are (in general) complex

Of course ☞ Physical solution is the real part of the above expression

This method of solution is only appropriate for linear differential equations

☞ if z is a complex variable

The method works because: &

☞ $\mathcal{L}$ some real linear differential operator which acts on this variable

$$\Downarrow$$

$$\mathrm{Re}[\mathcal{L}(z)] \equiv \mathcal{L}(\mathrm{Re}[z])$$

Damped Oscillatory Motion (cont'd)

Substituting $x = A\,\mathrm{e}^{-\mathrm{i}\,\omega\,t}$ in the equation of motion leads to

$$A\left[-\omega^2 - \mathrm{i}\,2\,\beta\,\omega + \omega_0^2\right]\mathrm{e}^{-\mathrm{i}\,\omega\,t} = 0$$

which reduces to the following quadratic equation for ω

$$\omega^2 + \mathrm{i}\,2\,\beta\,\omega - \omega_0^2 = 0$$

The solution to this equation is ☞ $\omega_\pm = -\mathrm{i}\,\beta \pm \sqrt{\omega_0^2 - \beta^2}$

Most general physical solution to damped oscillatory motion

$$x(t) = \mathrm{Re}\left[A_+\,\mathrm{e}^{-\mathrm{i}\,\omega_+\,t} + A_-\,\mathrm{e}^{-\mathrm{i}\,\omega_-\,t}\right]$$

$A_\pm$ are two arbitrary complex constants

We can distinguish three different cases

- ↗ **underdamped**
- ⇒ **critically damped**
- ↘ **overdamped**

Damped Oscillatory Motion (cont'd)

If $\beta < \omega_0$ ☞ the motion is said to be underdamped

The most general solution is written

$$x(t) = x_0\, e^{-\beta\, t} \cos(\omega_r\, t) + \left(\frac{v_0 + \beta\, x_0}{\omega_r}\right) e^{-\beta\, t} \sin(\omega_r\, t)$$

$$\omega_r = \sqrt{\omega_0^2 - \beta^2} \qquad x_0 = x(0) \qquad v_0 = dx(0)/dt$$

☞ oscillates at some real frequency ω_r
(somewhat less than natural frequency ω_0 of undamped system)

It can be seen that the solution

☞ decays exponentially in time
(at a rate proportional to the damping coefficient β)

Damped Oscillatory Motion (cont'd)

Phase diagram for underdamped motion

$$x(t) = Ae^{-\beta t}\cos(\omega_r t - \delta)$$

$$\dot{x}(t) = -Ae^{-\beta t}[\beta\cos(\omega_r t - \delta) + \omega_r \sin(\omega_r t - \delta)]$$

Changing variables

$$u = \omega_r x \qquad w = \beta x + \dot{x}$$

$$\Downarrow$$

$$u = \omega_r A e^{-\beta t}\cos(\omega_r t - \delta)$$

$$w = -\omega_r A e^{-\beta t}\sin(\omega_r t - \delta)$$

In polar coordinates

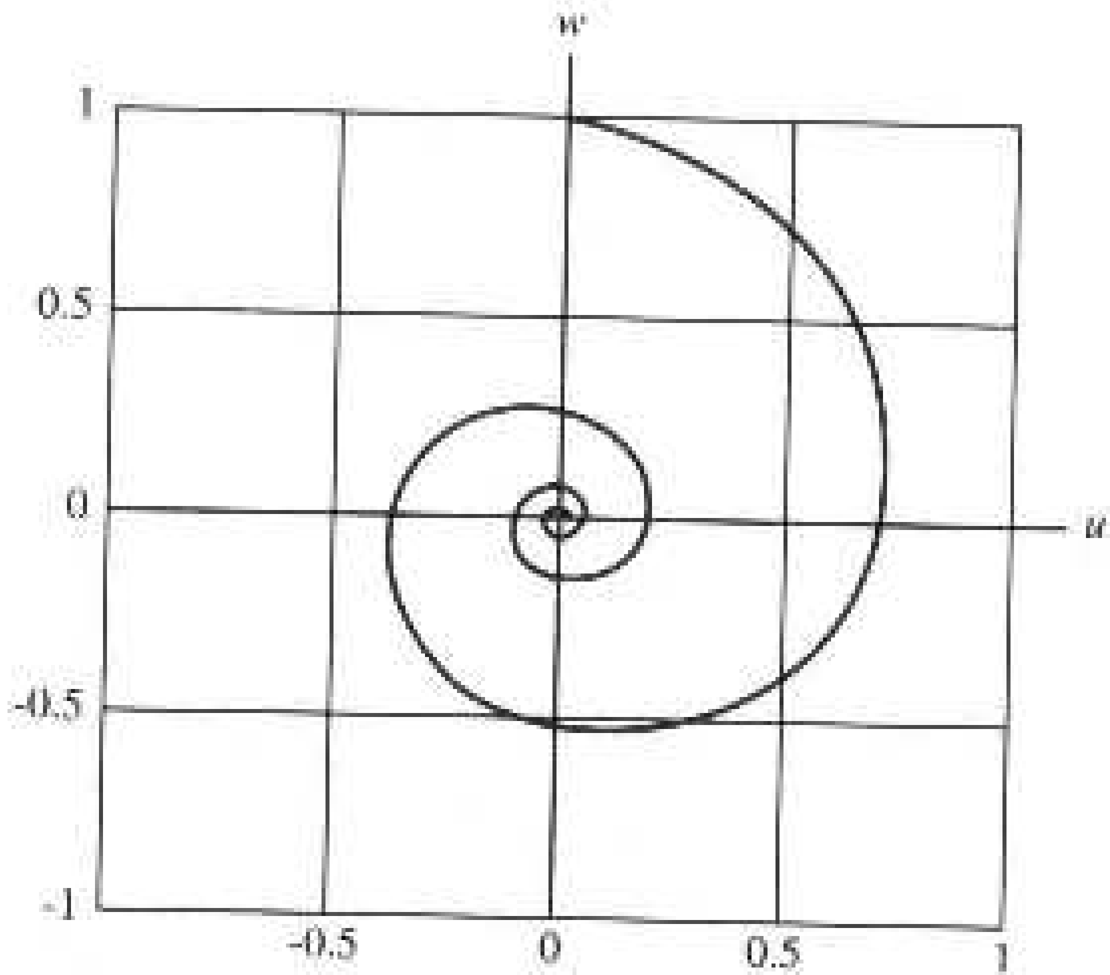
$$\rho = \sqrt{u^2 + w^2} \qquad \varphi = \omega_r t$$

$$\Downarrow$$

$$\rho = \omega_r A e^{-(\beta/\omega_r)\varphi}$$

Damped Oscillatory Motion (cont'd)

Equation of a logarithmic spiral



Damped Oscillatory Motion (cont'd)

If $\beta = \omega_0$ ☞ the motion is said to be critically damped

The most general solution is written

$$x(t) = [x_0\,(1 + \omega_0\,t) + v_0\,t]\,\mathrm{e}^{-\omega_0\,t}$$

- **The solution now decays without oscillating**

If $\beta > \omega_0$ ☞ the motion is said to be overdamped

The most general solution is written

$$x(t) = -\left(\frac{v_0 + \beta_-\,x_0}{\beta_+ - \beta_-}\right)\mathrm{e}^{-\beta_+\,t} + \left(\frac{v_0 + \beta_+\,x_0}{\beta_+ - \beta_-}\right)\mathrm{e}^{-\beta_-\,t}$$

$$\beta_\pm = \beta \pm \sqrt{\beta^2 - \omega_0^2}$$

- **The solution again decays without oscillating**
  (except there are now two independent decay rates)

Damped Oscillatory Motion (cont'd)

SUMMARY

The largest β_+ ☞ is always greater than the critically damped decay rate ω_0

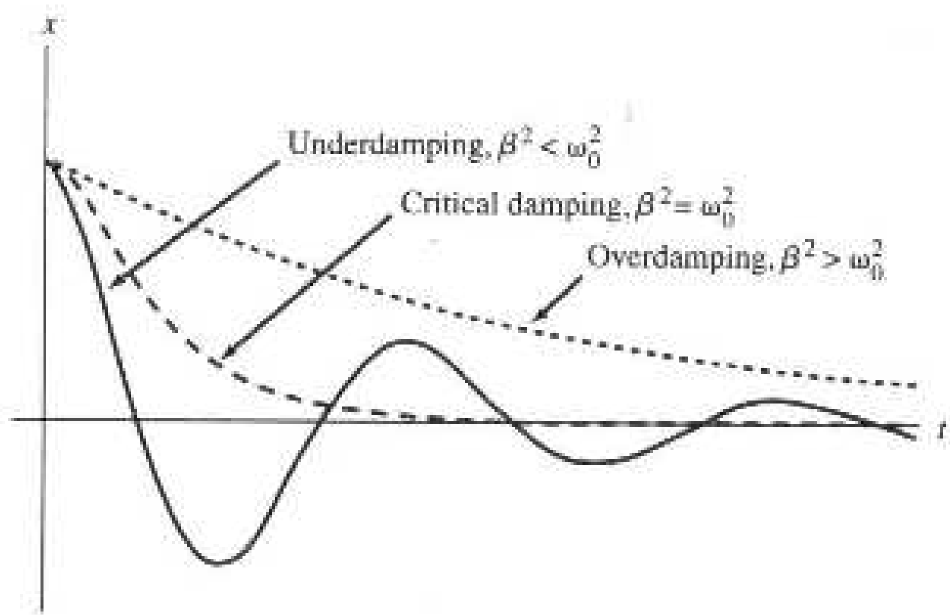
The smallest β_- ☞ is always less than this decay rate.

$\Downarrow$

☚ underdamped solution

☺ The critically damped solution is more rapidly damped than

☚ overdamped solutions



Sinusoidal Driving Forces

We saw that ☞

1-dimensional dynamical systems slightly perturbed from stable equilibrium point (and then left alone)
eventually return to this point at rate controlled by damping of the system

Now ☞

Suppose the same system is subject to an external force with fixed frequency ω

$\Downarrow$

system will eventually settle down to some steady oscillatory pattern of motion (with the same frequency)

Next ☞
Probe whether this is true by studying the properties of such a "driven oscillation"

Sinusoidal Driving Forces (cont'd)

Suppose that our system is subject to an external force of the form

$$f_{\rm ext}(t) = m\,\omega_0^2\,X_1\,\cos(\omega\,t)$$

X_1 ☚ typical ratio of the amplitude of external force to that of restoring force
Incorporating the external force into our perturbed equation of motion

$$\Downarrow$$

$$\frac{d^2x}{dt^2} + 2\,\beta\,\frac{dx}{dt} + \omega_0^2\,x = \omega_0^2\,X_1\,\cos(\omega\,t)$$

Trial Solution ☞ $\omega_0^2\,X_1\,\exp(-{\rm i}\,\omega\,t)$ ☚ ω is now a real parameter

again understood that physical terms are real parts of these expressions

$$a\left[-\omega^2 - {\rm i}\,2\,\beta\,\omega + \omega_0^2\right]\,{\rm e}^{-{\rm i}\omega\,t} = \omega_0^2\,X_1\,{\rm e}^{-{\rm i}\,\omega\,t}$$

$$\Downarrow$$

$$a = \frac{\omega_0^2\,X_1}{\omega_0^2 - \omega^2 - {\rm i}\,2\,\beta\,\omega}$$

Sinusoidal Driving Forces (cont'd)

In general ☚ a is a complex quantity $\Rightarrow a = D\, e^{i\,\delta}$
(D and δ are both real)

the physical solution takes the form ☞ $x(t) = D\,\cos(\omega\, t - \delta)$

$$\Downarrow$$

$$D = \frac{\omega_0^2\, X_1}{[(\omega_0^2 - \omega^2)^2 + 4\,\beta^2\,\omega^2]^{1/2}} \quad \text{and} \quad \delta = \tan^{-1}\left(\frac{2\,\beta\,\omega}{\omega_0^2 - \omega^2}\right)$$

Conclusion

In response to the applied sinusoidal force
the system executes a sinusoidal pattern of motion at the same frequency
with fixed amplitude D and phase-lag δ
(with respect to the external force)

Resonance Phenomena

Amplitude Resonance Frequency ☞ ω at which D is maximum

$$\left.\frac{dD}{d\omega}\right|_{\omega=\omega_{\rm R}} = 0$$

$$\Downarrow$$

$$\omega_{\rm R} = \sqrt{\omega_0^2 - 2\beta^2}$$

The resonance frequency $\omega_{\rm R}$ is lowered as the damping coefficient beta is increased

No resonance occurs if $\beta^2 > \omega_0^2/2$

($\omega_{\rm R}$ is imaginary and D decreases monotonically with increasing ω)

We customarily describe degree of damping in oscillating systems via quality factor

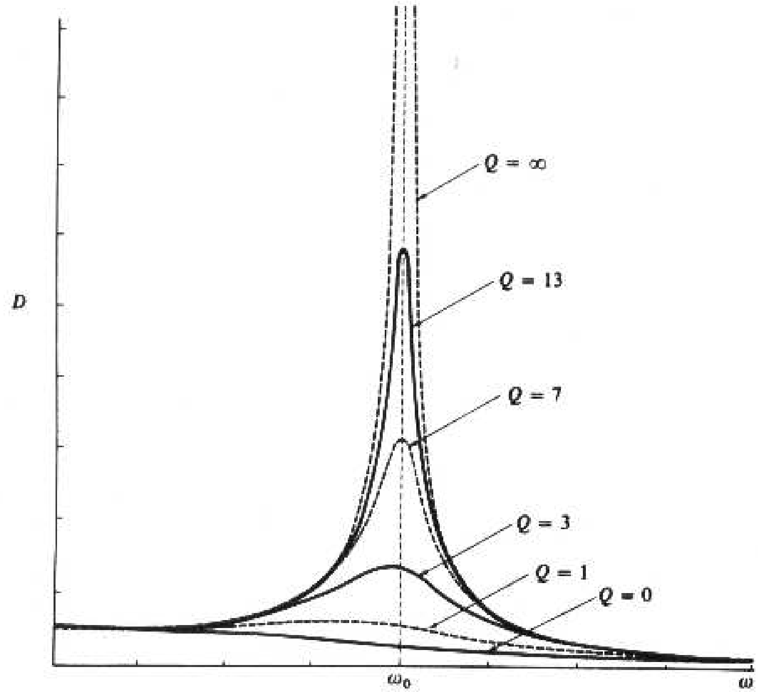
$$Q \equiv \frac{\omega_R}{2\beta}$$

- If little damping occurs ☞
$Q \gg 1$ and shape of the resonance curve approaches that of undamped oscillator

- Resonant condition is completely destroyed if damping is large and $Q \ll 1$

Resonance Phenomena (cont'd)

SUMMARY



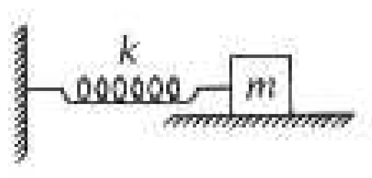
Electrical Oscillations

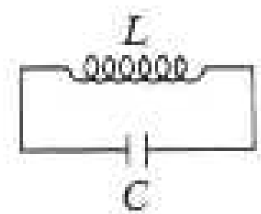
Consider the simple harmonic oscillator and the LC electrical circuit

- At some instant t
  - the charge on the capacitor C is $q(t)$
  - the current flowing through the inductor L is $I(t) = \dot{q}(t)$

Applying Kirchhoff's equation leads to a voltage drops around the circuit

$$L\,\frac{dI}{dt} + \frac{1}{C}\int I dt = 0$$





$$m\ddot{x} + kx = 0 \qquad\qquad L\ddot{q} + q/C = 0$$

$$\omega_0 = \sqrt{k/m} \qquad\qquad \omega_0 = 1/\sqrt{LC}$$

$$x(t) = x_0 \cos(\omega_0 t) \qquad\qquad q(t) = q_0 \cos(\omega_0 t)$$

Electrical Oscillations (cont'd)

Differentiating the expression for $q(t)$ ☞ $\dot{q} = I = -\omega_0 q_0 \sin(\omega_0 t)$

Squaring q and I ☞ $\frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{q_0^2}{C} = \text{constant}$

The term $LI^2/2$ represents the energy stored in the inductor
(corresponding to mechanical kinetic energy)

The term $q^2/2C$ represents the energy stored in the capacitor
(corresponding to the mechanical potential energy)

The sum of these two energies is constant ☚ the system is conservative

Analogous Mechanical and Electrical Quantities

Mechanical	Electrical
x ☚ displacement	q ☚ charge
$\dot{x}$ ☚ velocity	$\dot{q} = I$ ☚ current
m ☚ mass	L ☚ inductance
$2m\beta$ ☚ damping resistance	R ☚ resistance
k^{-1} ☚ mechanical compliance	C ☚ capacitance
$f_{\rm ext}$ ☚ amplitude of impressed force	$\mathcal{E}$ ☚ amplitude of impressed emf