Classical Mechanics

Fundamental aspects of Newton's theory of motion

☞ Newton's Laws ✓
☞ Inclined Plane ✓
☞ Projectiles ✓
☞ Conservation Theorems ✓
☞ Rocket Motion
☞ Motion in a General 1-dimensional Potential

Rocket Motion

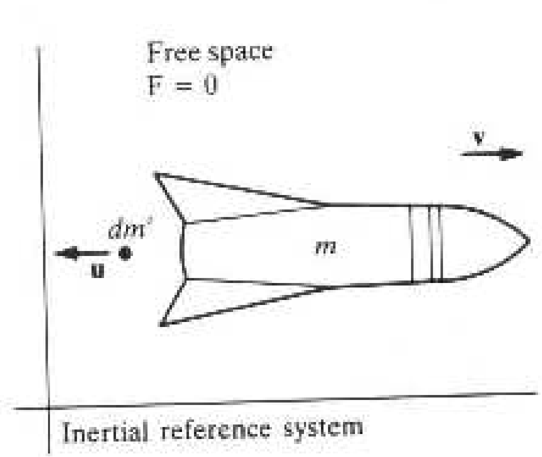
Interesting application of elementary Newtonian dynamics

I. Rocket motion in free space
☛ requires an application of the conservation of linear momentum

II. Vertical ascent of rockets under gravity
☛requires a more complicated application of Newton's Second Law

Rocket Motion in Free Space

- Assume that the space ship moves under the influence of no external forces
- Consider a closed system in which Newton's Second Law can be applied
- In outer space
  Motion of the space ship must depend enterely on its own energy
  It moves by the reaction of ejecting mass at high velocities
  To conserve $\vec{p}$ the space ship will have to move in the opposite direction



- the instantaneous total mass of the space ship is m

At some time t

- the instantaneous speed is v with respect to an inertial reference system

Rocket Motion in Free Space (cont'd)

Since motion is in x-direction we eliminate vector notation

During a time interval dt a positive mass dm' is ejected from the rocket engine (with a speed $-u$ with respect to the space ship)

Immediately after the mass dm' is ejected

- ☛ the speed is $v + dv$
- ☛ the mass is $m - dm'$

Initial momentum ☞ $p_{\rm initial} = mv$ (at time t)
Final momentum ☞ $p_{\rm final} = (m - dm')(v + dv) + dm'(v - u)$ (at time $t + dt$)

Conservation of linear momentum requires ☞ $p_{\rm initial} = p_{\rm final}$

$$p(t) = p(t + dt)$$

$$mv = (m - dm')(v + dv) + dm'(v - u)$$

$$mv = mv + mdv - vdm' - dm'dv + vdm' - udm'$$

Neglecting terms $\mathcal{O}(dv \times dm')$ ☞ $mdv = udm'$

$$dv = u\frac{dm'}{m}$$

Rocket Motion in Free Space (cont'd)

We considered dm' to be a positive mass ejected from the space ship

$\Downarrow$

The change in mass of the rocket itself is $dm = -dm'$

$\Downarrow$

$$dv = -u\frac{dm}{m}$$

Denoting: $m_0 \equiv$ initial mass and $v_0 \equiv$ initial speed ☚ integration leads to

$$\int_{v_0}^{v} dv = -u \int_{m_0}^{m} \frac{dm}{m}$$

$$v - v_0 = u \ln \left[\frac{m_0}{m}\right]$$

$$v = v_0 + u \ln \left[\frac{m_0}{m}\right]$$

Minimum mass (less fuel) of spaceship is limited by structural material
If fuel container is jettisoned after fuel has been burned

$\Downarrow$

mass of remaning spaceship is less

Rocket Motion in Free Space (cont'd)

Terminal speed limited by the ratio m_0/m ☚ multistage rockets

$m_0 =$ initial total mass of the ship
$m_1 = m_a + m_b$
$m_a =$ mass of the first-stage payload
$m_b =$ mass of the 1st-stage fuel containers
$v_1 =$ terminal speed of 1st-stage @ burnout
$v_1 = v_0 + u \ln \left[\frac{m_0}{m_1}\right]$

$m_a =$ initial total mass of the ship
$m_2 = m_c + m_d$
$m_b =$ mass of the second-stage payload
$m_d =$ mass of the 2nd-stage fuel containers
$v_1 =$ terminal speed of 2nd-stage @ burnout
$v_2 = v_1 + u \ln \left[\frac{m_a}{m_2}\right]$

$$v_2 = v_0 + u \ln \left[\frac{m_0\, m_a}{m_1\, m_2}\right]$$

The product of $(m_0 m_a/m_1 m_2)$ can be made much larger than m_0/m_1

Multi-stage rockets are commonly used in ascent under gravity

Spacecraft is propelled as a result of conservation of linear momentum

$$\downarrow$$

Engineers refer to the force term as rocket "thrust"

$$m\frac{dv}{dt} = -u\frac{dm}{dt} \equiv \text{Thrust}$$

Vertical Ascent Under Gravity

Motion of rockets attempting to leave the Earth's gravitational field
☛ quite complicated

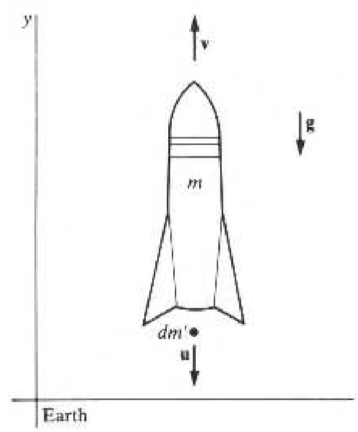
For analytical purposes here we assume:

(i). the rocket has only vertical motion
(ii). the air resistance is negligible
(iii). the acceleration of gravity is constant with height
(iv). the burn rate of the fuel is constant

The rocket's equation of motion is

$$F_{\rm ext} = -mg = \frac{d}{dt}(mv)$$

$$F_{\rm ext} = d(mv) = dp = p(t+dt) - p(t)$$



Vertical Ascent Under Gravity (cont'd)

$$p(t+dt) - p(t) = m\,dv + u\,dm$$

$$-mg\ dt = m\ dv + u\ dm$$

$$-mg = m\dot{v} + u\dot{m}$$

The fuel burn is constant ☛ $\dot{m} = -\alpha$ with $\alpha > 0$

$$\Downarrow$$

$$dv = \left(-g + \frac{\alpha}{m} u\right) dt$$

Equation with 3 unknowns $(v,\ m,\ t)$ ↑ ☛ but because of constant fuel burn rate

$$\Downarrow$$

$$dv = \left(\frac{g}{\alpha} - \frac{u}{m}\right) dm$$

Vertical Ascent Under Gravity (cont'd)

Taking initial $v = 0$ and initial mass $= m_0$

$$\int_0^v dv = \int_{m_0}^m \left(\frac{g}{\alpha} - \frac{u}{m}\right) dm$$

$$v = -\frac{g}{\alpha}(m_0 - m) + u \ln\left[\frac{m_0}{m}\right]$$

from fuel burn rate equation

$\int_{m_0}^m dm = -\alpha \int_0^t dt$ ☚ $m_0 - m = \alpha t$

$$\Downarrow$$

$$v = -gt + u \ln\left[\frac{m_0}{m}\right]$$

If the exhaust velocity u is not sufficiently to make $v > 0$ the rocket would remain on the ground

Saturn V: America's Moon Rocket

Saturn V -developed at NASA's Marshall Space Flight Center- was the largest in a family of liquid-propellant rockets that solved the problem of getting to the Moon

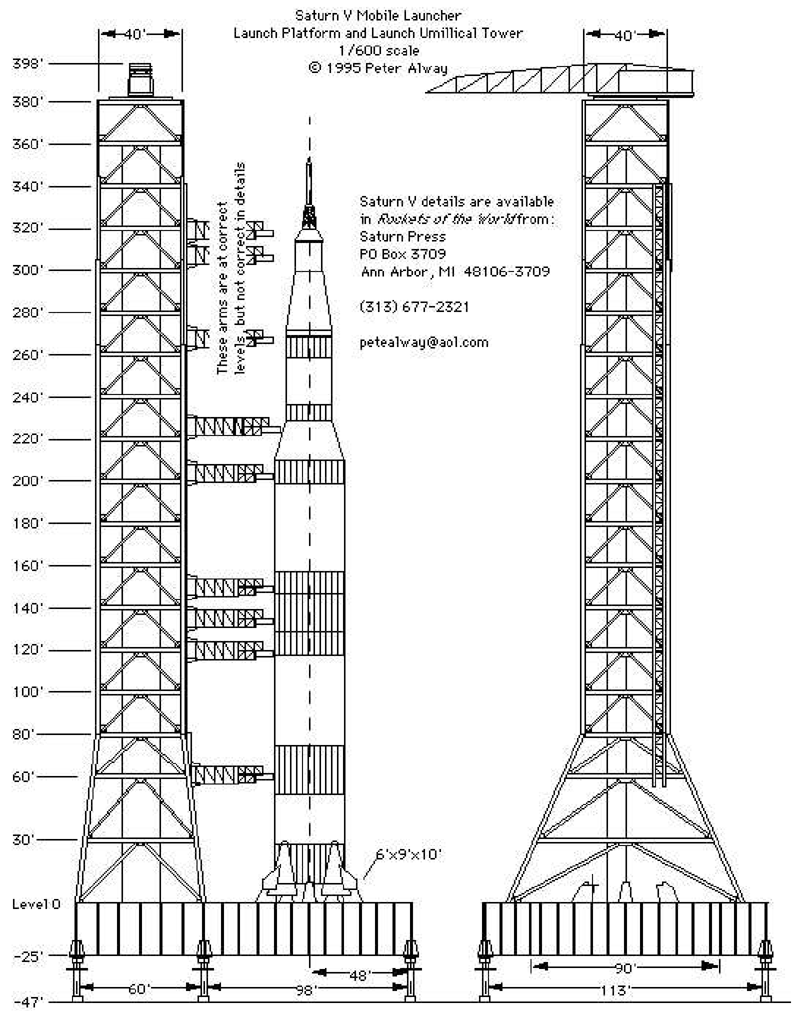
32 Saturns were launched ☞ not one failed!!!

The Saturn V was flight-tested twice without crew ❁ the first manned Saturn V sent the Apollo 8 astronauts into orbit around the Moon in December 1968
After two more missions to test the lunar landing vehicle ☞ in July 16th 1969 a Saturn V launched the crew of Apollo 11 to the first manned landing on the Moon

↑ Apollo 11 rocket blasts off

Estimate the final speed of the first stage at burnout and its vertical height

Saturn V: America's Moon Rocket (cont'd)



$m_0 \approx 2.8 \times 10^6$ kg
mean thrust $\approx 37 \times 10^6$ N

mass of first stage fuel $\approx 2.1 \times 10^6$ kg
$u \approx 2600$ m/s

Saturn V: America's Moon Rocket (cont'd)

Use definition of thrust to determine the fuel burn rate

$$\frac{dm}{dt} = \frac{\text{thrust}}{-u} \approx \frac{37 \times 10^6 \text{ N}}{-2600 \text{ m/s}} = -1.42 \times 10^4 \text{ kg/s}$$

The final rocket mass ☞ $m = 2.8 \times 10^6 \text{ kg} - 2.1 \times 10^6 \text{ kg} = 0.7 \times 10^6 \text{ kg}$

The speed of the space ship at burnout

$$v_\text{b} \approx -\frac{9.8 \text{ m/s}^2 \; 2.1 \times 10^6 \text{ kg}}{1.42 \times 10^4 \text{ kg/s}} + 2600 \text{ m/s} \;\; \ln \left[\frac{2.8 \times 10^6 \text{ kg}}{0.7 \times 10^6 \text{ kg}}\right]$$

$$v_\text{b} \approx 2.16 \times 10^3 \text{ m/s}$$

Time to burnout

$$t_\text{b} = \frac{m_0 - m}{\alpha} \approx \frac{2.1 \times 10^6 \text{ kg}}{1.42 \times 10^4 \text{ kg/s}} = 148 \text{ s}$$

t_b about two and a half minutes

Saturn V: America's Moon Rocket (cont'd)

$$v = \frac{dy}{dt} = -gt + u \ln \left[\frac{m_0}{m}\right]$$

$$\int dy = \int \left\{ -gt + u \ln \left[\frac{m_0}{m}\right] \right\} dt$$

Since $dm/dt = -\alpha$ ☞ $dt = -dm/\alpha$

$$y + \mathcal{C} = -\frac{1}{2} g t^2 - \frac{u}{\alpha} \int \ln \left[\frac{m_0}{m}\right] dm$$

using

$$\int \ln \left(\frac{a}{x}\right) dx = x \left[1 + \ln \left(\frac{a}{x}\right)\right]$$

$$\Downarrow$$

$$y + \mathcal{C} = -\frac{1}{2} g t^2 - \frac{u}{\alpha} \left[m + m \ln \left(\frac{m_0}{m}\right)\right]$$

Saturn V: America's Moon Rocket (cont'd)

Evaluate $\mathcal{C}$ from initial conditions @ $t = 0$ ☛ $y = 0$ and $m = m_0$

$$\mathcal{C} = -\frac{u m_0}{\alpha}$$

$$y_{\rm b} = u t_{\rm b} - \frac{1}{2} g t_{\rm b}^2 - \frac{m u}{\alpha} \ln\left[\frac{m_0}{m}\right]$$

$$y_{\rm b} \approx 2600~{\rm m/s}\,148~{\rm s} - \frac{1}{2}\,9.8~{\rm m/s}^2~148~{\rm s} - \frac{0.7\times 10^6~{\rm kg}\,2600~{\rm m/s}}{1.42\times 10^4~{\rm kg/s}} \times \ln\left[\frac{2.8\times 10^6~{\rm kg}}{0.7\times 10^6~{\rm kg}}\right]$$

$$y_{\rm b} \approx 9.98\times 10^4~{\rm m} \approx 100~{\rm km}$$

Motion in 1-dimensional Potentials

Consider a particle of mass m moving in the x-direction
– say, under the action of some x-directed force $f(x)$ –

Suppose that f is a conservative force ☜ e.g. gravity

It is convenient to specify f in terms of its associated potential energy

$$\Downarrow$$

potential energy of the object at position x
$\uparrow$

$$f(x) = -\frac{dU(x)}{dx}$$

- **We know that the total mechanical energy is a constant of motion**

$$T(x) = E - U(x)$$

- **We also know that a kinetic energy can never be negative**

(since neither m nor v^2 can be negative)

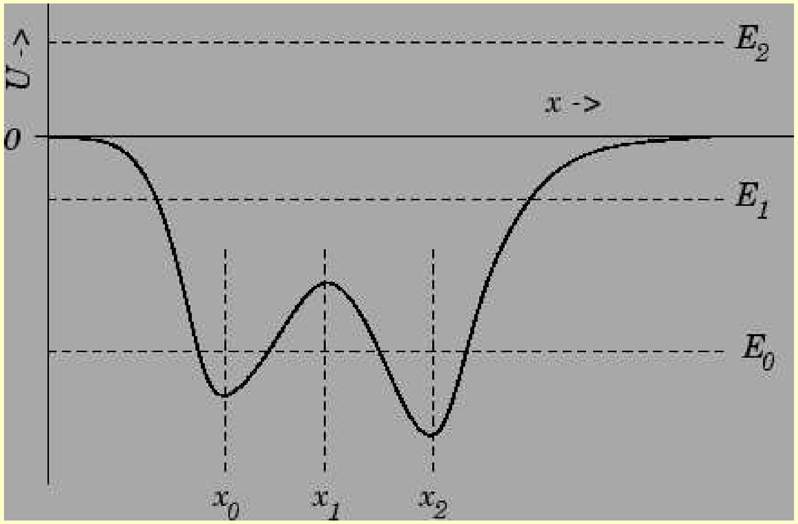
$$\Downarrow$$

motion of the particle restricted to the region
where the potential energy $U(x)$ falls below the value E

Motion in 1-dimensional Potentials (cont'd)

Consider the potential energy curve $U(x)$

(of some particle moving in a one-dimensional conservative force-field)



↑ example ↓

The gravitational potential energy of a cyclist freewheeling in a hilly region

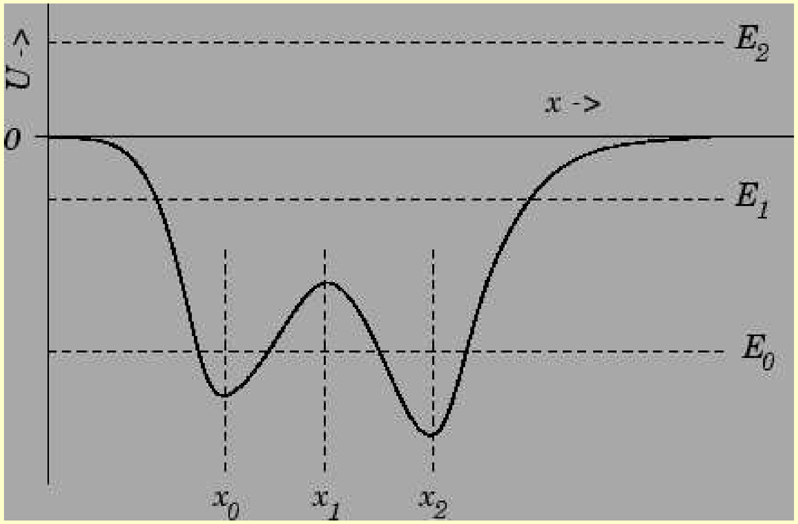
☞ Note that we have set the potential energy at infinity to zero

This is a useful (and quite common) convention

(recall that potential energy is undefined to within an arbitrary additive constant)

What can we deduce about the motion of the particle in this potential?

Motion in 1-dimensional Potentials (cont'd)



☛ Suppose that the total energy of the system is E_0

⇓

The particle is trapped inside one or other of the two dips in the potential (these dips are generally referred to as potential wells)

☛ Suppose that we now raise the energy to E_1

⇓

The particle is free to enter or leave each of the potential wells but its motion is still bounded to some extent since it cannot move off to infinity

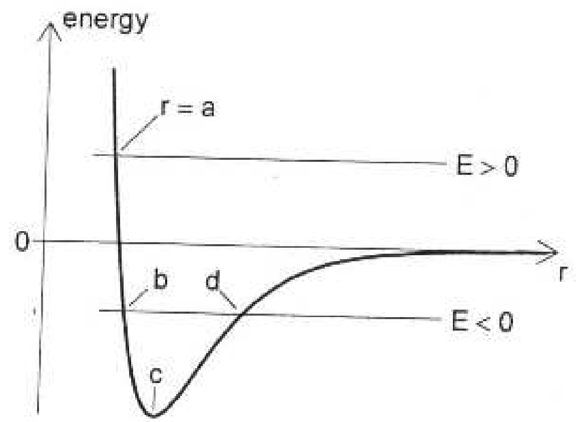
☛ Let's finally raise the energy to E_2

⇓

The particle is unbounded ☞ it can move off to infinity

Motion in 1-dimensional Potentials (cont'd)

Potential energy for typical diatomic molecules
(plotted as a function of the distance r between the 2 atoms)



☛ **bounded systems are characterized by $E < 0$**

If the potential energy at infinity is zero:

☛ **unbounded systems are characterized by $E > 0$**

☛ If $E > 0$
$\Rightarrow$ the 2 atoms cannot approach closer than the turning point $r = a$
but they can move apart to infinity
☛ If $E < 0$
$\Rightarrow$ the 2 atoms are trapped between the turning points at b and d
they form a bound molecule with eliptical orbits
☛ Equilibrium point $\rightarrow$ minimum energy (circular orbit with radius $r = c$)

Equilibrium States

SUMMARY

☞ becomes less bounded as E of the system increases

The motion of a particle moving in a potential

☞ becomes more bounded as E of the system decreases

⇓

If the energy becomes sufficiently small

⇓

system will settle down in some equilibrium state where the particle is stationary

How can we identify any prospective equilibrium states?

If the mass remains stationary then it must be subject to zero force (otherwise it would accelerate)

⇓

Equilibrium state characterized by

$$\frac{dU}{dx} = 0$$

Equilibrium States (cont'd)

Equilibrium states correspond to either a maximum or a minimum of $U(x)$

☛ **stable equilibrium points**

Distinction between:

☛ **unstable equilibrium points**

☛ When the system is slightly perturbed from a stable equilibrium point $\longrightarrow$ resultant force f should always be attempting to return the system to this point

$$\Downarrow$$

the equilibrium point $x = x_0$ is stable $\Leftrightarrow \left.\frac{df}{dx}\right|_{x_0} < 0$

stability $\Leftrightarrow$ the force acts on the opposite direction to the perturbation

the equilibrium point $x = x_0$ is unstable $\Leftrightarrow \left.\frac{df}{dx}\right|_{x_0} > 0$

In other words

☛ **stable equilibrium points corresponds to minima of $U(x)$**
☛ **unstable equilibrium points correspond to maxima of $U(x)$**

Equilibrium States (cont'd)

Definitions make perfect sense if $\mathrm{U(x)}$ is the gravitational potential

$\Downarrow$

U is directly proportional to height

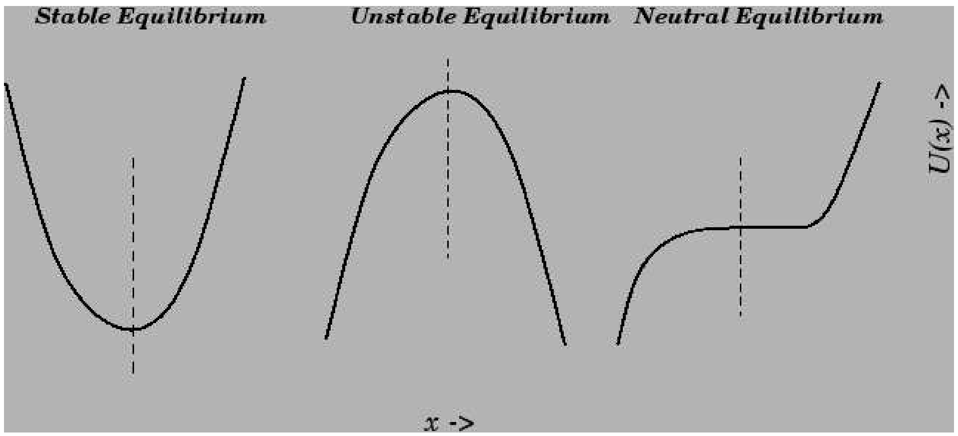
It is easy to confine a low energy particle at the bottom of a valley but very difficult to balance the same particle on the top of a hill

(since any slight perturbation to the particle will cause it to fall down the hill)

If $\left.\frac{dU}{dx}\right|_{x=x_0} = \left.\frac{d^2U}{dx^2}\right|_{x=x_0} = 0 \Rightarrow x = x_0$ is a neutral equilibrium point

We can move the particle slightly away from x_0 but will still remain in equilibrium (it will neither attempt to return to its initial state, nor will it continue to move).

SUMMARY



Double Well Potential

Consider the 1-dimensional potential

$$U(x) = \frac{-W d^2(x^2 + d^2)}{x^4 + 8d^4}$$

performing the change of variable $y = x/d$

$$Z(y) = \frac{U(x)}{W} = -\frac{(y^2+1)}{y^4+8}$$

Search for maxima and minima

$$\frac{dZ}{dy} = -\frac{2y}{y^4+8} + \frac{4y^3(y^2+1)}{(y^4+8)^2} = 0$$

after a bit of algebra

$$y(y^4 + 2y^2 - 8) = 0$$

$$y(y^2+4)(y^2-2) = 0 \Rightarrow y_0^2 = 2,\, 0$$

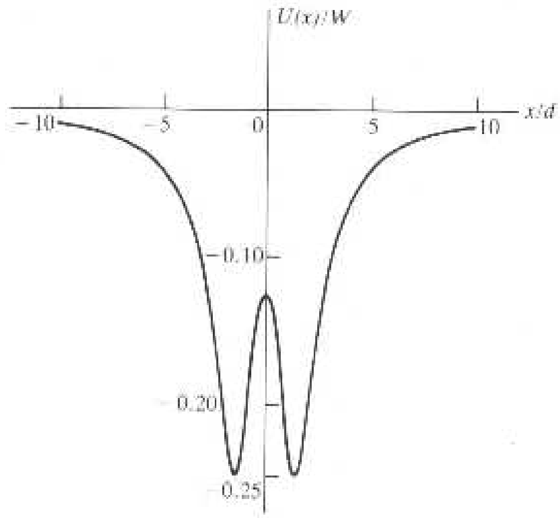
$$\Downarrow$$

$$x_{0_1} = 0 \qquad x_{0_2} = \sqrt{2}\,d \qquad x_{0_3} = -\sqrt{2}\,d$$

Double Well Potential (cont'd)

The equilibrium is stable at x_{0_2} and x_{0_3} but unsatble at x_{0_1}

The motion is bounded for all energies $E < 0$



We determine the turning points for $E = -W/8$

$$E = -\frac{W}{8} = U(y) = -\frac{W(y^2+1)}{y^4+8}$$

$$y^4 + 8 = 8y^2 + 8 \Rightarrow y^4 = 8y^2 \Rightarrow y = \pm 2\sqrt{2},\, 0$$

Turning points for $E = -W/8$ are $x_1 = -2\sqrt{2}\,d$ $x_2 = 2\sqrt{2}\,d$ and $x_3 = 0$

Spontaneous Symmetry Breaking

- Spontaneous symmetry breaking arises when a system that is symmetric with respect to some symmetry group goes into a vacuum state that is not symmetric
  - discrete ☞ such as the space group of a crystal
- The symmetry group can be
  - continuous (Lie group) ☞ such as the rotational symmetry of space

A common illustration of this phenomenon is a ball sitting on top of a hill

$\Downarrow$

Though the ball is in a completely symmetric state it is not a stable one (the ball can easily roll down the hill)

At some point

$\Downarrow$

the ball spontaneously rolls down the hill in one direction or another

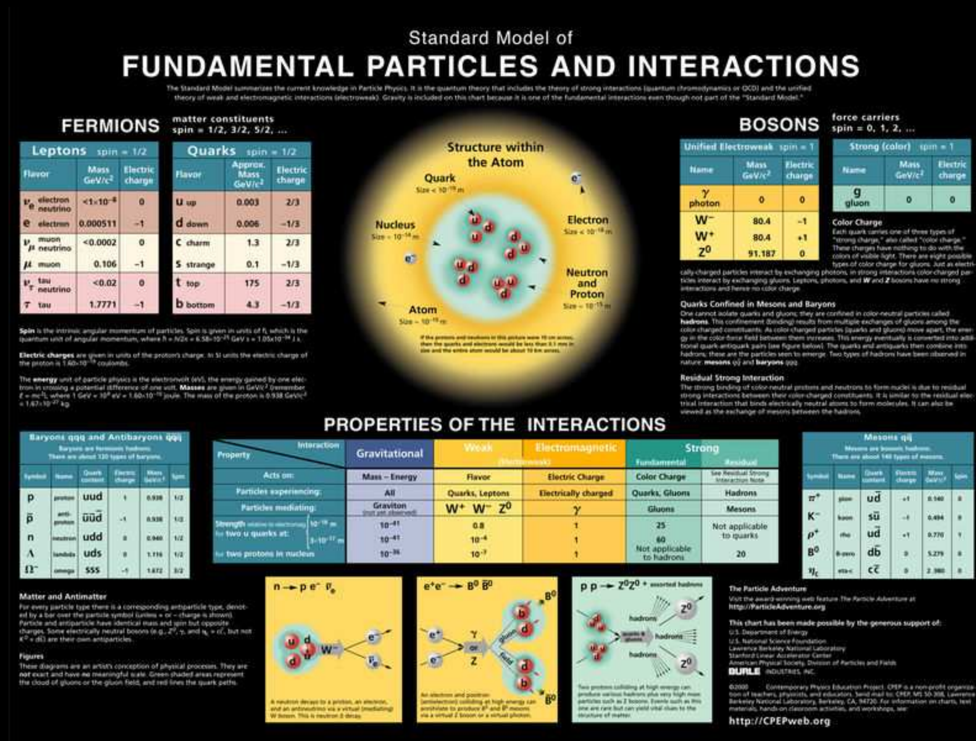
$\Downarrow$

The symmetry has been broken

$\Downarrow$

the direction the ball rolled down in has now been singled out from other directions

Elementary Particle Physics



SM masses ☜ Higgs mechanism

Homework

Study the properties of the Mexican hat potential

$$V(\phi) = (|\phi|^2 - v^2)^2$$

non-zero VEV of the Higgs field ☞ $v = 246$ GeV

spontaneously breaks the electroweak symmetry

1-dimensional Equation of Motion

Consider a particle moving in 1-dimension under the action of a conservative force

Since $T = \frac{1}{2} m v^2$ ☛ the energy conservation equation can be rearranged to give

$$v = \pm \left(\frac{2 [E - U(x)]}{m} \right)^{1/2}$$

$\pm$ signs correspond to motion to the left and to the right

Since $v = dx/dt$ this expression can be integrated

$$t = \pm \left(\frac{m}{2 E} \right)^{1/2} \int_{x_0}^{x} \frac{dx'}{\sqrt{1 - U(x')/E}}$$

with initial condition $x(t = 0) = x_0$

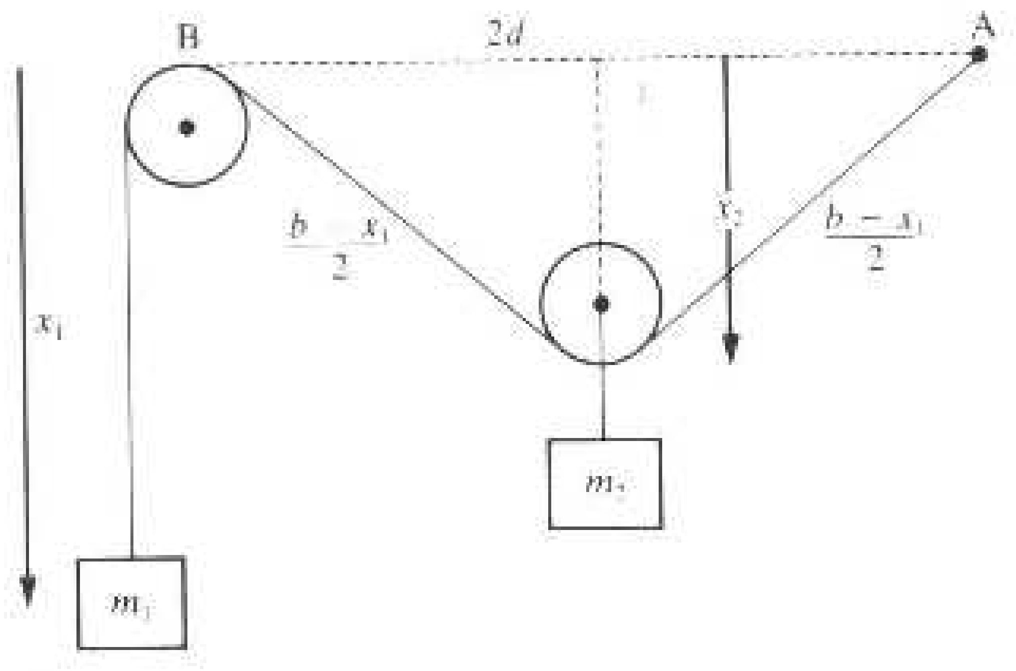
For sufficiently simple potential functions $U(x)$

$\Downarrow$

the above equation can be solved to give x as a function of t

Movable Pulley with lifting force directed downward

Consider the system of pulleys, masses, and string shown in the figure



A light string of length b is attached at point A, passes over a pulley at point B located a distance $2d$ away, and finally attaches to mass m_1.
Another pulley with mass m_2 attached passes over the string, pulling it down between A and B.
Calculate the distance x_1 when the system is in equilibrium and determine whether the equilibrium is stable or unstable.
The pulley are massless

Movable Pulley with lifting force directed downward (cont'd)

- We can solve the problem either using forces ($\ddot{x} = \dot{x} = 0$) or energy
- We chose energy because in equilibrium $T = 0$ ☞ analysis of potential energy
- We set $U = 0$ along the line AB

$$U = -m_1\, g\, x_1 - m_2\, g\, x_2$$

We assume the pulley holding the mass m_2 is small ☛ neglect pulley radius

$$x_2 = \sqrt{(b - x_1)^2/4 - d^2}$$

$$U = -m_1\, g\, x_1 - m_2\, g \sqrt{(b - x_1)^2/4 - d^2}$$

Equilibrium point $(x_1)_0 = x_0 \Leftrightarrow dU/dx_1 = 0$

$$\left.\frac{dU}{dx_1}\right|_0 = -m_1\, g + \frac{m_2\, g\, (b - x_0)}{4\sqrt{(b - x_0)^2/4 - d^2}} = 0$$

$$4\, m_1 \sqrt{(b - x_0)^2/4 - d^2} = m_2\, (b - x_0) ☛ (b - x_0)^2 (4m_1^2 - m_2^2) = 16 m_1^2 d^2$$

$$x_0 = b - \frac{4 m_1 d}{\sqrt{4m_1^2 - m_2^2}}$$

Real solution exists $\Leftrightarrow 4m_1^2 > m_2^2$

Movable Pulley with lifting force directed downward (cont'd)

Stability Analysis

$$\frac{d^2U}{dx_1^2} = \frac{-m_2\, g}{4\{[(b-x_1^2)/4] - d^2\}^{1/2}} + \frac{m_2\, g(b-x_1)^2}{16\{[(b-x_1^2)/4] - d^2\}^{3/2}}$$

Inserting $x_1 = x_0$

$$\left.\frac{d^2U}{dx_1^2}\right|_0 = \frac{g(4m_1 - m_2^2)^{3/2}}{4m_2^2\, d}$$

The condition for the equilibrium (real motion) ☛ $4m_1^2 > m_2^2$

$\Downarrow$

If it exists the equilibrium will be stable since $(d^2U/dx^2) > 0$